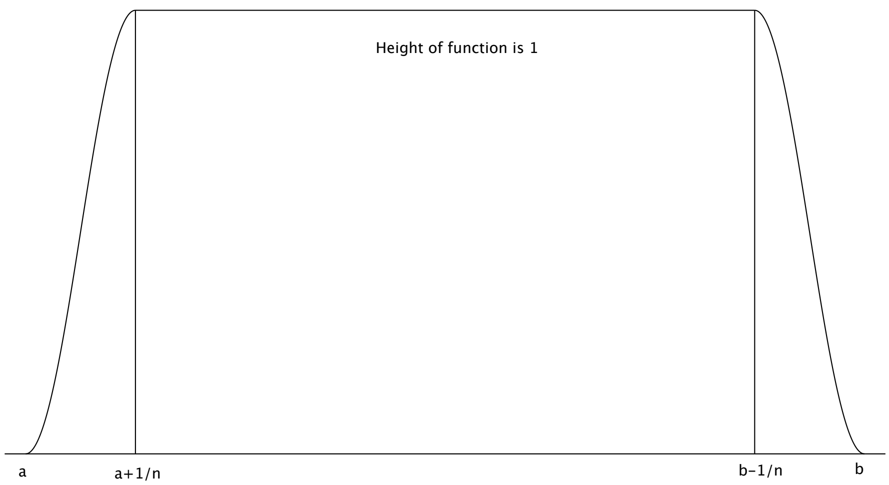
The elements of the Calculus of Variations.

We will assume the following.

Lemma 1. *For any interval* $[a, b]$ *and any positive integer with* $1/n < (b - a)/2$ *there is a smooth (in this setting this means the first derivative exists and is continuous) function,* $\phi_{a,b,n}(t)$ *that looks like*



That is $\phi_{a,b,n}(t)$ *is zero at the endpoints* $t = a$ *and* $t = b$ *of the interval, has the value* 1 *for* $a + 1/n \le t \le b - 1/n$ *and* $0 \le \phi_{a,b,n}(t) \le 1$. □

If you don't want to assume this you can check that

$$\phi_{a,b,n}(t) = \begin{cases} \sin^2(n(t-a)\pi/2), & a \le t \le a + 1/n; \\ 1, & a + 1/n \le t \le b - 1/n; \\ \sin^2(n(b-t)\pi/2), & b - 1/n \le t \le b \end{cases}$$

does the trick.

Proposition 2. *Let* $f\colon [a, b] \to \mathbf{R}$ *be a continuous function with a continuous derivative on the interval* $[a, b]$ *such that for all functions,* g*, that are continuous functions with continuous derivatives on* $[a, b]$ *and with* $g(a) = g(b) = 0$ *that*

$$\int_a^b f(t)g(t)\,dt = 0.$$

Then $f(t) = 0$ *for all* $t \in [a, b]$.

Problem 1. Prove this. *Hint:* Let for any n the function $g_n(t) = \phi_{a,b,n}(t)f(t)$ is continuous with continuous derivative and $g_n(a) = g_n(b) = 0$. Therefore

$$\int_a^b f(t)g_n(t)\,dt = 0.$$

What happens when we take the limit as $n \to \infty$ in this?

Now let $L(t,x,y)$ be a smooth (this time smooth means that the first and second partial derivatives exist and are continuous) functions of (t,x,y). We call such a function a ***Lagrangian*** Let $u\colon [a,b]\to \mathbf{R}$ be a function such that for all other functions v on $[a,b]$ that

(1)

$$u(a)=v(a),\ u(b)=v(b) \implies \int_a^b L(t,u(t),\dot u(t))\,dt \le \int_a^b L(t,v(t),\dot v(t))\,dt.$$

That is u minimizes $\int_a^b L(t,u(t),\dot u(t))\,dt$ over all functions with the same boundary values as u.

Our goal is to show that this implies any such minimizer will satisfy a certain differential equation. Toward this end let $g\colon [a,b]\to \mathbf{R}$ be any smooth function with

$$g(a)=g(b)=0.$$

Then for any real number ε the function

$$u_\varepsilon(t)=u(t)+\varepsilon g(t)$$

has $u_\varepsilon(a)=u(a)$ and $u_\varepsilon(b)=u(b)$. Therefore if we define a function of ε by

$$f(\varepsilon)=\int_a^b L(t,u_\varepsilon(t),\dot u_\varepsilon(t))\,dt$$

and note that when $\varepsilon=0$ that $u_0=u$ we see that f has a minimum at $\varepsilon=0$. Therefore the derivative of f at $\varepsilon=0$ vanishes. That is

$$\begin{aligned} f'(0) &= \frac{d}{d\varepsilon}\int_a^b L(t,u_\varepsilon(t),\dot u_\varepsilon(t))\,dt\bigg|_{\varepsilon=0} \\ &= \int_a^b \frac{d}{d\varepsilon}L(t,u_\varepsilon(t),\dot u_\varepsilon(t))\bigg|_{\varepsilon=0}\,dt \\ &= 0 \end{aligned} \tag{2}$$

where we can move the derivative under the integral by a theorem of advanced calculus.

Problem 2. Use the chain rule to show that

$$\frac{d}{d\varepsilon}L(t,u_\varepsilon(t),\dot u_\varepsilon(t))\bigg|_{\varepsilon=0} = \frac{\partial L}{\partial x}(t,u(t),\dot u(t))g(t)+\frac{\partial L}{\partial y}(t,u(t),\dot u(t))\dot g(t).$$

(Here the notation is that $L=L(t,x,y)$ is a function of (t,x,y). Thus $\frac{\partial L}{\partial x}$ is the partial derivative with respect to the second variable and $\frac{\partial L}{\partial y}$ the derivative with respect to the third. It is also common to write these as $\frac{\partial L}{\partial u}$ and $\frac{\partial L}{\partial \dot u}$.)

Using this in the equation (2) we get

$$
\begin{aligned}
0 &= \int_a^b \left(\frac{\partial L}{\partial x}(t, u(t), \dot{u}(t))g(t) + \frac{\partial L}{\partial y}(t, u(t), \dot{u}(t))\dot{g}(t) \right) dt \\
&= \int_a^b \frac{\partial L}{\partial x}(t, u(t), \dot{u}(t))g(t)dt + \int_a^b \frac{\partial L}{\partial y}(t, u(t), \dot{u}(t))\dot{g}(t)dt
\end{aligned}
\tag{3}
$$

Problem 3. Use integration by parts and that $g(a) = g(b) = 0$ to show

$$\int_a^b \frac{\partial L}{\partial y}(t, u(t), \dot{u}(t))\dot{g}(t)dt = -\int_a^b \left(\frac{d}{dt}\frac{\partial L}{\partial y}(t, u(t), \dot{u}(t)) \right) g(t)\, dt.$$

Problem 4. Combine the last problem with equation (3) to conclude

$$\int_a^b \left(\frac{d}{dt}\frac{\partial L}{\partial y}(t, u(t), \dot{u}(t)) - \frac{\partial L}{\partial x}(t, u(t), \dot{u}(t)) \right) g(t)\, dt = 0$$

for all smooth g on $[a, b]$ with $g(a) = g(b) = 0$.

We can now state the main result.

Theorem 3. *Let u be a smooth function that solves the minimization problem* (1) *above. Then u satisfies the **Euler-Lagrange** equation*

$$\frac{d}{dt}\frac{\partial L}{\partial y}(t, u(t), \dot{u}(t)) - \frac{\partial L}{\partial x}(t, u(t), \dot{u}(t)) = 0.$$

Problem 5. Prove this.

Problem 6. Let $L(t, x, y) = \sqrt{1 + y^2}$ and let the interval be $[a, b] = [0, 1]$. Then for a function $u(t)$ on these interval

$$\int_0^1 L(t, u(t), \dot{u}(t))\, dt = \int_0^1 \sqrt{1 + \dot{u}(t)^2}\, dt$$

is just the length of the graph of u. Find the Euler-Lagrange equation for this integral and show that the solutions are straight lines.

Definition 4. The Lagrangian is ***time independent*** if and only if $L = L(x, y)$ does not depend on t. If L is time independent define the energy to be

$$E = y\frac{\partial L}{\partial y} - L$$

□

Theorem 5 (Conservation of energy)**.** *Assume the Lagrangian, L, is time independent and that u is a solution to the Euler-Lagrange equation for L. Then the **energy***

$$E(t) = \dot{u}\frac{\partial L}{\partial y}(\dot{u}(t), u(t))$$

is constant.

Proof. It is enough to show that $E'(t) = 0$.

$$\begin{aligned} E'(t) &= \frac{d}{dt}\left(\dot{u}(t)\frac{\partial L}{\partial y}(\dot{u}(t), u(t)) - L(\dot{u}(t), u(t))\right) \\ &= \ddot{u}(t)\frac{\partial L}{\partial y}(\dot{u}(t), u(t)) + \dot{u}(t)\frac{d}{dt}\frac{\partial L}{\partial y}(\dot{u}(t), u(t)) \\ &\qquad - \frac{\partial L}{\partial y}(\dot{u}(t), u(t))\ddot{u}(t) - \frac{\partial L}{\partial x}(\dot{u}(t), u(t))\dot{u}(t) \\ &= \dot{u}(t)\left(\frac{d}{dt}\frac{\partial L}{\partial y}(\dot{u}(t), u(t)) - \frac{\partial L}{\partial x}(\dot{u}(t), u(t))\dot{u}(t)\right) \\ &= 0 \end{aligned}$$

where at the last step we used the Euler-Lagrange equation. □

One reason law of conservation of energy is nice is that it reduces solving the second order Euler-Lagrange equation to solving a first order equation $E =$ constant. Let us look at an example. If $y = f(x)$ is revolved around the x-axis with $f(x) > 0$ and $a \leq x \leq b$ is revolved around the x axis, then the area of the resulting surface is

$$A = 2\pi \int_a^b f(x)\sqrt{1 + f'(x)^2}\, dx$$

So to find the surfaces of revolution of least area (a problem we will come back to) we wish to find the solutions to the Euler-Lagrange equations for the Lagrangian

$$L(y, x) = x\sqrt{1 + y^2}.$$

If $u(t)$ is a solution to the Euler-Lagrange equation then (using what I hope is transparent notation) the energy is

$$\begin{aligned} E &= \dot{u}\frac{\partial}{\partial \dot{u}}\left(u\sqrt{1 + \dot{u}^2}\right) - u\sqrt{1 + \dot{u}^2} \\ &= \dot{u}\left(\frac{u\dot{u}}{\sqrt{1 + \dot{u}}}\right) - u\sqrt{1 + \dot{u}^2} \\ &= u\left(\frac{\dot{u}^2}{\sqrt{1 + \dot{u}^2}} - \sqrt{1 + \dot{u}^2}\right) \end{aligned}$$

Multiply by $\sqrt{1 + \dot{u}^2}$

$$E\sqrt{1 + \dot{u}^2} = u\left(\dot{u}^2 - 1 - \dot{u}^2\right) = -u^2.$$

Squaring gives

$$E^2(1 + \dot{u}^2) = u^2$$

and then solve for $\dot{u}$

$$\dot{u} = \frac{\sqrt{u^2 - E^2}}{E}$$

Now we can just check directly that

$$u(t) = E\cosh((t - t_0)/E)$$

is a solution. This is also the shape of a hanging chain.