Mathematics 551 Homework, January 9, 2024

We start by reviewing some vector algebra. Let $\mathbf{a} = (a_1, a_2)$ and $\mathbf{b} = (b_1, b_2)$ be vectors in $\mathbb{R}^2$ and c a scalar. Then the sum of $\mathbf{a}$ and $\mathbf{b}$ is

$$\mathbf{a} + \mathbf{b} = (a_1 + b_1, a_2 + b_2)$$

and the product of $\mathbf{a}$ by the c is

$$c\mathbf{a} = (ca_1, cb_1).$$

It is common to use the notation

$$\mathbf{i} = (1, 0), \qquad \mathbf{j} = (0, 1).$$

With this notation we can write $\mathbf{a}$ as

$$\mathbf{a} = a_1\mathbf{i} + a_2\mathbf{j}.$$

The ***inner product*** of $\mathbf{a}$ and $\mathbf{b}$ is

$$\mathbf{a} \cdot \mathbf{b} = a_1 b_1 + a_2 b_2.$$

Then

$$\mathbf{a} \cdot \mathbf{a} = (a_1)^2 + (a_2)^2$$

which is the square of the length of $\mathbf{a}$. We use the notation

$$\|\mathbf{a}\| = \sqrt{\mathbf{a} \cdot \mathbf{a}}$$

for the ***length*** of $\mathbf{a}$.

If c_1 and c_2 are scalars and $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ are vectors are scalars then the inner product has the following properties:

- $\mathbf{b} \cdot \mathbf{w} = \mathbf{w} \cdot \mathbf{v}$.
- $(c_1\mathbf{u} + c_2\mathbf{v}) \cdot \mathbf{w} = c_1\mathbf{u} \cdot \mathbf{w} + c_2\mathbf{v} \cdot \mathbf{w}$.

Consequences of these that will come up are

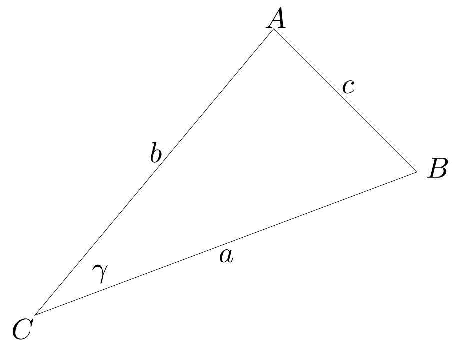
$$\begin{aligned} \|\mathbf{a} + \mathbf{b}\| &= \|\mathbf{a}\|^2 + 2\mathbf{a} \cdot \mathbf{b} + \|\mathbf{a}\|^2 \\ \|\mathbf{a} - \mathbf{b}\| &= \|\mathbf{a}|^2 - 2\mathbf{a} \cdot \mathbf{b} + \|\mathbf{b}\|^2. \end{aligned}$$

Problem 1. Prove these formulas. □

A very important property of the inner product is given by

Theorem 1. *If $\mathbf{a}$ and $\mathbf{b}$ are nonzero vectors and θ is the angle between them, then*

$$\mathbf{a} \cdot \mathbf{b} = \|\mathbf{a}\| \|\mathbf{b}\| \cos(\theta). \tag{1}$$



Problem 2. In the triangle shown a, b, and c are the side lengths of $\triangle ABC$ and γ is the angle between $\overrightarrow{CB}$ and and $\overrightarrow{CA}$. Use Theorem 1 to show

$$c^2 = a^2 + b^2 - 2ab\cos(\gamma).$$

Hint: Let $\mathbf{u} = \overrightarrow{CB}$, $\mathbf{v} = \overrightarrow{CA}$ and $\mathbf{w} = \overrightarrow{AB}$. Then $\mathbf{w} = \mathbf{u} - \mathbf{v}$ and therefore $\|\mathbf{w}\|^2 = \|\mathbf{u} - \mathbf{v}\|^2$. □

A Corollary of Theorem 1 is that non-zero vectors $\mathbf{a}$ and $\mathbf{b}$ are perpendicular if and only if $\mathbf{a} \cdot \mathbf{b} = 0$.

Problem 3. Show that for any vectors $\mathbf{a}$ and $\mathbf{b}$ that $\mathbf{a} + \mathbf{b}$ and $\mathbf{a} - \mathbf{b}$ are perpendicular if and only if $\|\mathbf{a}\| = \|\mathbf{b}\|$. □

Problem 4. Define a map $J\colon \mathbb{R}^2 \to \mathbb{R}^2$ by

$$J(x, y) = (-y, x).$$

Show for all non-zero vectors $\mathbf{v}$ that

(a) $\|J\mathbf{v}\| = \|\mathbf{v}\|$
(b) $\mathbf{v}$ and $J\mathbf{v}$ are always perpendicular. □

Problem 5. Let for any $\theta \in \mathbb{R}$ let and $\mathbf{v} \in \mathbb{R}^2$ let

$$R(\theta)\mathbf{v} = \cos(\theta)\mathbf{v} + \sin(\theta)J\mathbf{v}.$$

Show

(a) $\|R(\theta)\mathbf{v}\| = \|\mathbf{v}\|$
(b) The angle between $\mathbf{v}$ and $R(\theta)\mathbf{v}$ is θ.

Remark 2. For those of you who have had linear algebra the map $R(\theta)$ is linear and has matrix

$$\begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix}.$$

As you may have seen in your linear algebra class, this is just the matrix form of a rotation by . □

We now recall some calculus for functions $\mathbf{c}\colon [a, b] \to \mathbb{R}^2$ or more generally functions $\mathbf{c}\colon [a, b] \to \mathbb{R}^n$ where $\mathbb{R}^n$ is the space of n-tuples $(x_1, x_2, \ldots, x_n)$. We give the formulas for $n = 2$, and generalizing to higher dimensions is easy. First if

$$\mathbf{c}(t) = (x(t), y(t))$$

then the ***derivative*** of $\mathbf{c}(t)$ is

$$\mathbf{c}'(t) = (x'(t), y'(t)).$$

That is computing the derivative of the function $\mathbf{c}(t) = (x(t), y(t))$ is the same as computing the derivative of each component. The official definition is in terms of a limit;

$$\mathbf{c}'(t) = \lim_{h \to 0} \frac{1}{h} \left(\mathbf{c}(t+h) - \mathbf{c}(t) \right).$$

With this definition some of the results of single variable calculus carry over without much change.

Theorem 3. *Let $\mathbf{f}, \mathbf{g} \colon [a,b] \to \mathbb{R}^2$ be differentiable vector valued functions and define a scalar valued function by*

$$h(t) = \mathbf{f}(t) \cdot \mathbf{g}(t).$$

Then the product rule

$$h'(t) = \mathbf{f}'(t) \cdot \mathbf{g}(t) + \mathbf{f}(t) \cdot \mathbf{g}'(t)$$

holds. That is

$$\big(\mathbf{f}(t) \cdot \mathbf{g}(t)\big)' = \mathbf{f}'(t) \cdot \mathbf{g}(t) + \mathbf{f}(t) \cdot \mathbf{g}'(t).$$

Problem 6. Prove this. *Hint:* Reduce this to the one variable product as follows. Let $\mathbf{f}(t) = (f_1(t), f_2(t))$ and $\mathbf{g}(t) = (g_1(t), g_2(t))$. Then

$$\mathbf{f}(t)\mathbf{g}(t) = f_1(t)g_1(t) + f_2(t)g_2(t).$$

Then

$$\big(\mathbf{f}(t)\mathbf{g}(t)\big)' = \big(f_1(t)g_1(t)\big)' + \big(f_2(t)g_2(t)\big)'$$

You can now use the one variable product rule on the terms $\big(f_1(t)g_1(t)\big)'$ and $\big(f_1(t)g_2(t)\big)'$ and rearrange the results to get the desired formula. $\square$

Corollary 4. *Let $\mathbf{f} \colon [a,b] \to \mathbb{R}^2$ be differentiable. Then*

$$\frac{d}{dt} \|\mathbf{f}(t)\|^2 = 2\mathbf{f}(t) \cdot \mathbf{f}'(t).$$

and at points where $\mathbf{f}(t) \neq \mathbf{0}$

$$\frac{d}{dt} \|\mathbf{f}(t)\| = \frac{\mathbf{f}(t)}{\|\mathbf{f}(t)\|} \cdot \mathbf{f}'(t).$$

Problem 7. Prove this. *Hint:* For the first formula let $\mathbf{g} = \mathbf{f}$ in Theorem 3. For the second use $\|\mathbf{f}(t)\| = \big(\|\mathbf{f}(t)\|^2\big)^{\frac{1}{2}}$. $\square$

Here is anther product rule.

Proposition 5. *Let $\mathbf{f} \colon [a,b] \to \mathbb{R}^2$ be a differentiable vector valued function and let $h \colon [a,b] \to \mathbb{R}$ be a differentiable scalar valued function. Then*

$$\frac{d}{dt} \left(h(t)\mathbf{f}(t) \right) = h'(t)\mathbf{f}(t) + h(t)\mathbf{f}'(t).$$

Problem 8. Prove this. $\square$

We now integrate vector valued functions. Let $\mathbf{f}\colon [a,b] \to \mathbb{R}^2$ be continuous and write it as

$$\mathbf{f}(t) = (f_1(t), f_2(t)).$$

Then

$$\int_a^b \mathbf{f}(t)\,dt = \left(\int_a^b f_1(t)\,dt, \int_a^b f_2(t)\,dt \right).$$

That is integrating a vector valued function is the same as integrating each of its component functions.

Proposition 6. *Let $\mathbf{f}\colon [a,b] \to \mathbb{R}^2$ be a vector valued function and $\mathbf{a}$ a constant vector. Then*

$$\mathbf{a} \cdot \int_a^b \mathbf{f}(t)\,dt = \int_a^b \mathbf{a} \cdot \mathbf{f}(t)\,t.$$

Problem 9. Prove this. *Hint:* Write $\mathbf{f}(t) = (f_1(t), f_2(t))$ and $\mathbf{a} = (a_1, a_2)$ and expand each side of the equation to be proven in terms of the definitions. $\square$

Using that $\cos(\theta) \le 1$ for all θ we see that Equation (1) implies

$$\mathbf{a} \cdot \mathbf{b} \le \|\mathbf{a}\| \|\mathbf{b}\|$$

for all vectors $\mathbf{a}$ and $\mathbf{b}$. This is the ***Cauchy-Schwartz inequality***.

Theorem 7. *Let $\mathbf{f}\colon [a,b] \to \mathbb{R}^2$ be a continuous vector valued function. Then the inequality*

$$\left\| \int_a^b \mathbf{f}(t)\,dt \right\| \le \int_a^b \|\mathbf{f}(t)\|\,dt$$

holds.

Problem 10. Prove this. *Hint:* If $\int_a^b \mathbf{f}(t)\,dt = \mathbf{0}$, then the result holds. So assume that $\int_a^b \mathbf{f}(t)\,dt \ne \mathbf{0}$. Now prove the result along the following lines.

(a) Let $\mathbf{a}$ be the vector

$$\mathbf{a} = \left\| \int_a^b \mathbf{f}(t)\,dt \right\|^{-1} \int_a^b \mathbf{f}(t)\,dt.$$

and show that $\mathbf{a}$ is a unit vector, that is

$$\|\mathbf{a}\| = 1$$

and that

$$\left\| \int_a^b \mathbf{f}(t)\,dt \right\| = \mathbf{a} \cdot \int_a^b \mathbf{f}(t)\,dt$$

(b) Now use Proposition 6 to show

$$\left\| \int_a^b \mathbf{f}(t)\,dt \right\| = \int_a^b \mathbf{a} \cdot \mathbf{f}(t)\,dt.$$

(c) Use that $\mathbf{a}$ is a unit vector and the Cauchy-Schwartz inequality to show

$$\mathbf{a} \cdot \mathbf{f}(t) \leq \|\mathbf{f}(t)\|.$$

(d) Put all these pieces together to complete the proof. $\square$

We now already have enough machinery to start proving some interesting results. Let $\mathbf{a}$ and $\mathbf{b}$ be vectors in the plane and let $r > 2\|\mathbf{a}-\mathbf{b}\|$. That is r is more than twice the distance between $\mathbf{a}$ and $\mathbf{b}$. Let

$$\mathcal{E} = \{\mathbf{x} \in \mathbb{R}^2 : \|\mathbf{x}-\mathbf{a}\| + \|\mathbf{x}-\mathbf{b}\| = 2r\}$$

be the set of all points, $\mathbf{x}$, such that the sum of he distances from $\mathbf{x}$ to the two points $\mathbf{a}$ and $\mathbf{b}$ is constant. As you most likely already know, this is the ellipse with foci $\mathbf{a}$ and $\mathbf{b}$. Let $\mathbf{c}(t)$ be a curve that moves on $\mathcal{E}$. That is $\mathbf{c}(a,b) \to \mathbb{R}$ is a curve so that for all t

$$\|\mathbf{c}(t)-\mathbf{a}\| + \|\mathbf{c}(t)-\mathbf{b}\| = 2r. \tag{2}$$

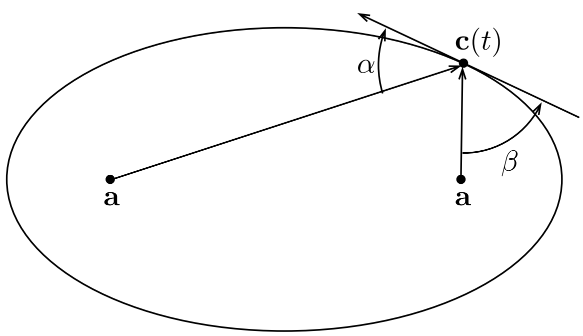


Figure 1. An ellipse with focal points $\mathbf{a}$ and $\mathbf{b}$. The line at $\mathbf{c}(t)$ is tangent to $\mathcal{E}$ at $\mathbf{c}(t)$ and points in the direction of the velocity vector $\mathbf{c}'(t)$.

Problem 11. In Figure 1 show that the angles α and β are equal. *Hint:* To do this is we use a trick that will come up at least 42 more times this term. This is to take an equation we know and take derivatives of the equation to get new equations. In many cases these new equations can be interrupted geometrically. In this case take the derivative of Equation (2) and show the result can be rewritten as

$$\mathbf{c}'(t) \cdot \frac{\mathbf{c}(t)-\mathbf{a}}{\|\mathbf{c}(t)-\mathbf{a}\|} = -\mathbf{c}'(t) \cdot \frac{\mathbf{c}(t)-\mathbf{b}}{\|\mathbf{c}(t)-\mathbf{b}\|}$$

and that this implies $\alpha = \beta$. $\square$

Problem 12. Let $\mathbf{a}$ and $\mathbf{b}$ be points in the plane and let

$$\mathcal{H} = \{\mathbf{x} : \|\mathbf{x}-\mathbf{a}\| - \|\mathbf{x}-\mathbf{b}\| = 2r\}$$

where r is a constant with $r > \|\mathbf{a}-\mathbf{b}\|$. This is one branch of a hyperbola. Let $\mathbf{c}(t)$ move on this hyperbola as in Figure 2. Use an argument similar

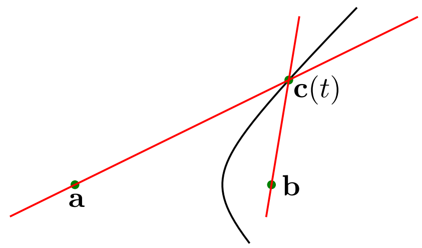


FIGURE 2. The hyperbola $\|\mathbf{x} - \mathbf{a}\| - \|\mathbf{x} - \mathbf{b}\| = 2r$. The tangent to this hyperbola at $\mathbf{c}(t)$ bisects the angle between the lines through $\mathbf{c}(t)$ and $\mathbf{a}$ and $\mathbf{b}$.

to that in Problem 11 to show for the tangent line to $\mathcal{H}$ as $\mathbf{c}(t)$ bisects the angle between the lines $\overleftrightarrow{\mathbf{ac}(t)}$ and $\overleftrightarrow{\mathbf{bc}(t)}$. $\square$