Mathematics 551 Homework, January 17, 2024

Here is a summary of part of the plot to date. If $\mathbf{c}\colon [a,b] \to \mathbb{R}^2$ is a unit speed curve, that is $\|\mathbf{c}'(s)\| = 1$ for all s, the unit tangent is

$$\mathbf{t}(s) = \mathbf{c}'(s) = \left(\frac{dx}{ds}, \frac{dy}{ds}\right)$$

where $\mathbf{c}$ has the coordinate repersentation

$$\mathbf{c}(s) = (x(s), y(s)).$$

Then the unit normal is

$$\mathbf{n}(s) = \left(-\frac{dy}{ds}, \frac{dx}{ds}\right).$$

Taking the derivative of the equation

$$\mathbf{t} \cdot \mathbf{t} = 1$$

with respect to s gives

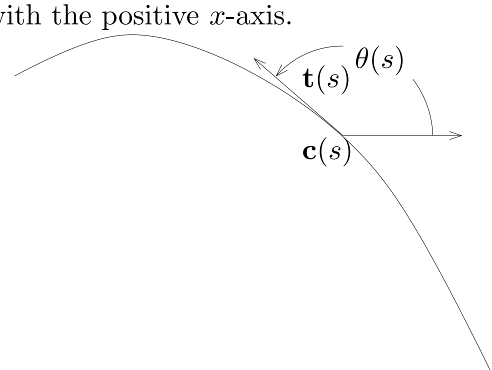
$$2\mathbf{t} \cdot \frac{d\mathbf{t}}{ds} = 0.$$

Therefore $\dfrac{d\mathbf{t}}{ds}$ is a scalar multiple of $\mathbf{n}$. That is there is a scaler function $\kappa(s)$ such that

$$\frac{d\mathbf{t}}{ds} = \kappa(s)\mathbf{n}(s).$$

This function is the ***curvature*** of $\mathbf{c}$.

We can give a somewhat more geometric description of κ. Let $\theta(s)$ be the angle $\mathbf{t}(s)$ makes with some fixed vector. To be concrete let it be the angle that $\mathbf{t}(s)$ makes with the positive x-axis.



Then the unit tangent and normal are

$$\mathbf{t}(s) = (\cos(\theta(s)), \sin(\theta(s))), \qquad \mathbf{n}(s) = (-\sin(\theta(s)), \cos(\theta(s)))$$

Then we can use the chain rule to compute

$$\frac{d\mathbf{t}}{ds} = \frac{d\theta}{ds}(-\sin(\theta(s)), \cos(\theta(s))) = \frac{d\theta}{ds}\mathbf{n}(s).$$

Comparing our two formulas for the derivative $\dfrac{d\mathbf{t}}{ds}$ shows that

$$\kappa = \frac{d\theta}{ds}.$$

Therefore the curvature is the rate of change of the direction of motion (as measured by the angle) with respect to arc length.

We have computed the derivatives of $\mathbf{c}$ and $\mathbf{t}$. We now want to compute the derivative of $\mathbf{n}$. In the current setting probably the most natural way is to use that

$$\mathbf{n}(s) = (-\sin(\theta(s)), \cos(\theta(s)))$$

and taking the derivative gives

$$\frac{d\mathbf{n}}{ds} = \frac{d\theta}{ds}(-\cos(\theta(s)), -\sin(\theta(s))) = -\kappa(s)\mathbf{t}(s).$$

When we look at curves in $\mathbb{R}^3$ we will need a different method to compute the derivative of the normal. Here is the idea. The vectors $\mathbf{t}$ and $\mathbf{n}$ are a basis for $\mathbb{R}^2$. Thus any other vector is a linear combination of these two:

$$\mathbf{v} = a\mathbf{t} + b\mathbf{n}.$$

And we can give formulas for a and b.

Proposition 1. *If $\mathbf{v} = a\mathbf{t} + b\mathbf{n}$, then*

$$a = \mathbf{v}\cdot\mathbf{t}, \qquad b = \mathbf{v}\cdot\mathbf{n}.$$

Problem 1. Prove this. *Hint:* We know (and you can assume) that the vectors $\mathbf{t}$ and $\mathbf{n}$ satisfy

$$\mathbf{t}\cdot\mathbf{t} = 1, \qquad \mathbf{t}\cdot\mathbf{n} = 0, \qquad \mathbf{n}\cdot\mathbf{n} = 1.$$

Now take $\mathbf{v} = a\mathbf{t} + b\mathbf{n}$ and take the dot product of both sides with $\mathbf{t}$ to see that $a = \mathbf{v}\dot{\mathbf{t}}$. Then take the dot product of both sides with $\mathbf{n}$ to get the formula for b. □

Using this we have

$$\frac{d\mathbf{n}}{ds} = \left(\frac{d\mathbf{n}}{ds}\cdot\mathbf{t}\right)\mathbf{t} + \left(\frac{d\mathbf{n}}{ds}\cdot\mathbf{n}\right)\mathbf{n}$$

To take the easiest of the terms first we take the derivative of $\mathbf{n}\cdot\mathbf{n} = 1$ to get

$$2\frac{d\mathbf{n}}{ds}\cdot\mathbf{n} = 0.$$

For the other term take the derivative of

$$\mathbf{n}\cdot\mathbf{t} = 0$$

to get

$$\frac{d\mathbf{n}}{ds}\cdot\mathbf{t} + \mathbf{n}\cdot\frac{d\mathbf{t}}{ds} = 0$$

which we rewrite as

$$\frac{d\mathbf{n}}{ds} \cdot \mathbf{t} = -\mathbf{n} \cdot \frac{d\mathbf{t}}{ds}.$$

But we know $\dfrac{d\mathbf{t}}{ds} = \kappa(s)\mathbf{n}$. Using this in the last displayed formula gives

$$\frac{d\mathbf{n}}{ds} \cdot \mathbf{t} = -\mathbf{n} \cdot \kappa(s)\mathbf{n} = -\kappa(s).$$

Putting these formulas together gives

$$\frac{d\mathbf{n}}{ds} = -\kappa(s)\mathbf{t}(s).$$

We summarize these calculations in the basic result:

Theorem 2 (Planar Frenet–Serret Formulas)**.** *For a unit speed curve* $\mathbf{c}\colon [a,b] \to \mathbb{R}^2$ *the formuals*

$$\begin{aligned} \frac{d\mathbf{c}}{ds} &= \mathbf{t} \\ \frac{d\mathbf{t}}{ds} &= \qquad \kappa(s)\mathbf{n} \\ \frac{d\mathbf{n}}{ds} &= -\kappa(s)\mathbf{t} \end{aligned}$$

hold where κ is the curvature of the curve. □

We now see that κ tells us about the curve.

Proposition 3. *Let* $\mathbf{c}\colon [a,b] \to \mathbb{R}^2$ *be a unit speed curve with* $\kappa \equiv 0$*. Then* $\mathbf{c}$ *is a part of a straight line.*

Problem 2. Prove this. *Hint:* From the Frenet-Serret we have

$$\frac{d\mathbf{t}}{ds} = \kappa\mathbf{n} = \mathbf{0}.$$

Thus the derivative of $\mathbf{t}$ is identically zero and therefore $\mathbf{t}$ is constant, say

$$\mathbf{t} = \mathbf{t}_0$$

for a constant vector. Using anther of the Frenet-Serret formulas we have

$$\frac{d\mathbf{c}}{ds} = \mathbf{t}_0.$$

Integrate this to get that $\mathbf{c}(s) = s\mathbf{t}_0 + \mathbf{c}_0$ for some constant vector $\mathbf{c}_0$. □

Since many curves do not come to use with a unit speed parameterization, and finding explicit unit speed parameterizations is hard work, we would like a formulas for curvature in terms of arbitrary parameterizations. Using the wedge product, $\mathbf{v}\wedge\mathbf{w}$, of vector makes these formulas easier to derive. Recall this is just a two dimensional version of the cross product: if $\mathbf{v} = (v_1, v_1)$ and $\mathbf{w} = (w_1, w_2)$ then

$$\mathbf{v} \wedge \mathbf{w} = \begin{vmatrix} v_1 & v_2 \\ w_1 & w_2 \end{vmatrix} = v_1 w_2 - v_2 w_1.$$

The properties of this product we will use are

$$\begin{aligned}
\mathbf{v} \wedge \mathbf{v} &= 0 \\
\mathbf{w} \wedge \mathbf{v} &= -\mathbf{v} \wedge \mathbf{w} \\
(a\mathbf{v}_1 + b\mathbf{v}_2) \wedge \mathbf{w} &= a\mathbf{v}_1 \wedge \mathbf{w} + b\mathbf{v}_2 \wedge \mathbf{w}.
\end{aligned}$$

where $\mathbf{v}, \mathbf{v}_1, \mathbf{v}_2$, and $\mathbf{w}$ are vectors and a and b are scalars. And also important for use are that for the unit tangent and normals to a curve

$$\begin{aligned}
\mathbf{t} \wedge \mathbf{t} = \mathbf{n} \wedge \mathbf{n} &= 0 \\
\mathbf{t} \wedge \mathbf{n} = -\mathbf{n} \wedge \mathbf{t} &= 1.
\end{aligned}$$

Problem 3. Verify the formula above for $\mathbf{t} \wedge \mathbf{n}$. *Hint:* There are many ways to do this. In terms of what we have done so far, maybe the easiest to to use that $\mathbf{t} = (\cos(\theta), \sin(\theta))$ and $\mathbf{n} = (-\sin(\theta), \cos(\theta))$. □

So let $\mathbf{c} \colon [a, b] \to \mathbb{R}^2$ be a curve with $\mathbf{c}'(t) \neq \mathbf{0}$ for all t, but which is not necessary unit speed. Let s be an arc length parameter along $\mathbf{c}$. Let

$$v = \|\mathbf{c}'(t)\| = \left\| \frac{d\mathbf{c}}{dt} \right\|$$

be the speed of $\mathbf{c}$. By the chain rule

$$\frac{d\mathbf{c}}{dt} = \frac{ds}{dt}\frac{d\mathbf{c}}{ds} = \frac{ds}{dt}\mathbf{t}.$$

As $\mathbf{t}$ is a unit vector this implies

$$\frac{ds}{dt} = \left\| \frac{d\mathbf{c}}{dt} \right\| = v.$$

So the velocity vector of $\mathbf{c}$ is

$$\frac{d\mathbf{c}}{dt} = v\mathbf{t}.$$

The acceleration vector is

$$\begin{aligned}
\frac{d^2\mathbf{c}}{dt^2} &= \frac{d}{dt}\frac{d\mathbf{c}}{dt} \\
&= \frac{d}{dt}(v\mathbf{t}) \\
&= \frac{dv}{dt}\mathbf{t} + v\frac{d\mathbf{t}}{dt} \\
&= \frac{dv}{dt}\mathbf{t} + v\frac{ds}{dt}\frac{d\mathbf{t}}{ds} \\
&= \frac{dv}{dt}\mathbf{t} + vv\kappa\mathbf{n} \\
&= \frac{dv}{dt}\mathbf{t} + v^2\kappa\mathbf{n}.
\end{aligned}$$

We now have

$$\begin{aligned}\frac{d\mathbf{c}}{dt}\wedge\frac{d^2\mathbf{c}}{dt^2} &= (v\mathbf{t})\wedge\left(\frac{dv}{dt}\mathbf{t}+v^2\kappa\mathbf{n}\right)\\ &= v^3\kappa(s) \qquad\qquad\qquad (\text{using } \mathbf{t}\wedge\mathbf{t}=0 \text{ and } \mathbf{t}\wedge\mathbf{n}=1).\end{aligned}$$

This gives a formula for κ

$$\kappa = \frac{1}{v^3}\left(\frac{d\mathbf{c}}{dt}\wedge\frac{d^2\mathbf{c}}{dt^2}\right).$$

If $\mathbf{c}(t)$ has the coordinate representation

$$\mathbf{c}(t) = (x(t), y(t))$$

then

$$\frac{d\mathbf{c}}{dt} = (x'(t), y'(t)), \qquad \frac{d^2\mathbf{c}}{dt^2} = (x''(t), y''(t))$$

and therefore the speed is

$$v = \left\|\frac{d\mathbf{c}}{dt}\right\| = \left(x'(t)^2+y'(t)^2\right)^{1/2}$$

and

$$\frac{d\mathbf{c}}{dt}\wedge\frac{d^2\mathbf{c}}{dt^2} = \begin{vmatrix} x'(t) & y'(t)\\ x''(t) & y''(t)\end{vmatrix} = x'(t)y''(t)-x''(t)y'(t).$$

This gives

Theorem 4. *For a C^2 curve $\mathbf{c}\colon [a,b]\to\mathbb{R}^2$ with $\mathbf{c}'(t)\neq\mathbf{0}$ for all t, the curvature is given by*

$$\kappa = \frac{x'(t)y''(t)-x''(t)y'(t)}{(x'(t)^2+y'(t)^2)^{3/2}}$$

□

We can now do a few more examples. A circle centered at the point (x_0, y_0) with radius r and transversed in the positive direction (that is counterclockwise) is parameterized by

$$\mathbf{c}(t) = (x_0 + r\cos(t), y_0 + r\sin(t)).$$

Problem 4. Show this circle has constant curvature $1/r$. Draw a picture showing that it is curving to the left (which is why the curvature is constant). □.

Now let's go around this circle in the opposite direction:

$$\mathbf{c}(t) = (x_0 + r\cos(t), y_0 - r\sin(t)).$$

Problem 5. Show this circle has constant curvature $-1/r$ and draw a picture showing that it is curving to the right. □

Here is anther way to derive the curvature formula which is probably more like what you did in your calculus class. We have seen that one formula for curvature is

$$\kappa = \frac{d\theta}{ds}.$$

We do our usual chain rule trick:

$$\kappa = \frac{dt}{ds}\frac{d\theta}{dt} = \frac{1}{v}\frac{d\theta}{dt}.$$

If $\mathbf{c} = (x(t), y(t))$ then the tangent vector is $\mathbf{c}'(t) = (x'(t), y'(t))$ and if θ is the angle this vector makes with the positive x-axis we have

$$\tan(\theta) = \frac{y'(t)}{x'(t)}$$

that is

$$\theta = \arctan\left(\frac{y'(t)}{x'(t)}\right).$$

Problem 6. Use the formula for the derivative of the arc tangent to expand

$$\kappa = \frac{1}{v}\frac{d\theta}{dt} = \frac{1}{v}\frac{d}{dt}\arctan\left(\frac{y'(t)}{x'(t)}\right)$$

and show that the result is

$$\kappa = \frac{x'(t)y''(t) - x''(t)y'(t)}{(x'(t)^2 + y'(t)^2)^{3/2}}$$

as before. □

Problem 7. Let $f\colon [a,b] \to \mathbb{R}^2$ be a function with second derivative continuous. Then $\c(t) = (t, f(t))$ parameterizes the graph of f. Use Theorem 4 to show the curvature is

$$\kappa(t) = \frac{f''(t)}{(1 + f'(t)^2)^{3/2}}.$$

Usually in this case we will just use $t = x$ as the parameter and write

$$\kappa(x) = \frac{f''(x)}{(1 + f'(x)^2)^{3/2}}$$

or just

$$\kappa = \frac{y''}{(1 + (y')^2)^{3/2}}$$ □

Problem 8. To continue the previous problem show the unit tangent and normal to the graph of $y = f(x)$ are

$$\mathbf{t}(x) = \frac{1}{\sqrt{1 + f'(x)^2}}(1, f'(x)), \quad \mathbf{n}(x) = \frac{1}{\sqrt{1 + f'(x)^2}}(-f'(x), 1)$$ □

We have seen in class that if the curvature is identically zero, the curve is part of a straight line. We would also like to know what happens if the curvature is an nonzero constant.

Theorem 5. *Let* $\mathbf{c}\colon [a,b] \to \mathbb{R}^2$ *be a unit speed curve that has constant curvature* $\kappa = \kappa_0 \neq 0$. *Then* $\mathbf{c}$ *is on a circle of radius* $r = 1/|\kappa_0|$.

Problem 9. Prove this along the following lines. Based on examples above it is reasonable to guess that if $\mathbf{c}$ moves on a circle, that the center of the circle is

$$\mathbf{P}(s) = \mathbf{c}(s) + \frac{1}{\kappa}\mathbf{n}.$$

Use the Frenet-Serret formulas to show

$$\frac{d\mathbf{P}}{ds} = \mathbf{0}$$

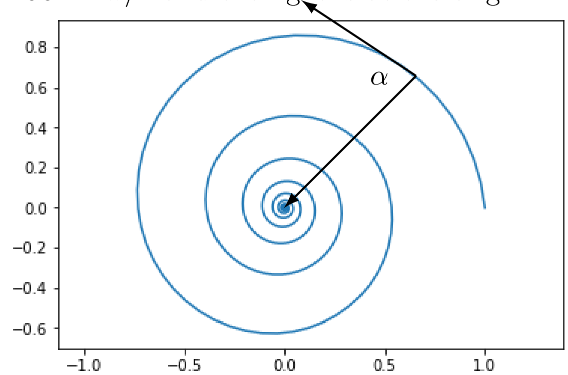
and therefore $\mathbf{P}(s) = \mathbf{P}_0$ is constant. Then show

$$\|\mathbf{c}(s) - \mathbf{P}_0\| = \frac{1}{|\kappa_0|}$$

to complete the proof. □

Now it is time to do some examples.

Problem 10. One theory about why moths fly into a bright light is that that are using the moon to navigate by keeping it at a angle constant to their direction of motion. This would keep them moving in a constant direction. But if there is a light that is brighter than the moon they mistake this for the moon and it the angle they are using is less that $\pi/2$ this leads to them spiraling into the light. The figure shows the case where the angle, α, is just a little less than $90^\circ = \pi/2$ and the light is at the origin.



If this were a differential equations class the problem would be given the angle, to find the curve. But we will start with a curve and show that it works. Let $a > 0$ and let

$$\mathbf{c}(t) = (e^{-at}\cos(t), e^{-at}\sin(t)) = e^{-at}(\cos(t), \sin(t)).$$

(a) Show that for this curve the angle, α, between $-\mathbf{c}(t)$ and $\mathbf{c}'(t)$ satisfies

$$\cos(\alpha) = \frac{a}{\sqrt{1+a^2}}$$

and therefore is constant.

(b) For this curve $\mathbf{c}(0) = (1, 0)$ and on the interval $[0, \infty)$ the spiral winds around the origin infinitely many times. Despite this the length of the curve is finite. Find the length.

(c) Compute the curvature of this curve. What happens to the curvature as $t \to \infty$? □

Problem 11*.* The ellipse with equation

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

has the parameterization

$$\mathbf{c}(t) = (a\cos(t), b\sin(t)).$$

This is not a unit speed parameterization. Use the formulas above to compute the unit tangent $\mathbf{t}(t)$, the unit normal, $\mathbf{n}(t)$, and the curvature $\kappa(t)$. □