Mathematics 551 Homework.

Contents

1. Calculations in polar coordinates.

First a trick or two to that make some calculations easier. Let

$$\mathbf{e}_1(\theta) = (\cos(\theta), \sin(\theta))$$
$$\mathbf{e}_2(\theta) = (-\sin(\theta), \cos(\theta))$$

These vectors are both unit vectors (that is have $\|\mathbf{e}_j\| = 1$, and are orthogonal (i.e. $\mathbf{e}_1 \cdot \mathbf{e}_2 = 0$). Also $\mathbf{e}_1 \wedge \mathbf{e}_2 = 1$. Also useful is that their derivatives are

$$\mathbf{e}_1' = \mathbf{e}_2$$
$$\mathbf{e}_2' = -\mathbf{e}_1.$$

Often writing curves in terms of this basis simplifies calculations it avoids a good deal (but not all) algebra and having to simplify expressions involving trigonometric functions. To be a little more explicit about how this works if we write two vectors $\mathbf{v}$ and $\mathbf{w}$ in the basis $\mathbf{e}_1$ and $\mathbf{e}_2$ as

$$\mathbf{v} = v_1\mathbf{e}_1 + v_2\mathbf{e}_2$$
$$\mathbf{w} = w_1\mathbf{e}_1 + w_2\mathbf{e}_2$$

Then the following hold:

$$\|\mathbf{v}\|^2 = v_1^2 + v_2^2$$
$$\|\mathbf{w}\|^2 = w_1^2 + w_2^2$$
$$\mathbf{v} \cdot \mathbf{w} = v_1w_1 + v_2w_2$$
$$\mathbf{v} \wedge \mathbf{w} = \begin{vmatrix} v_1 & v_2 \\ w_1 & w_2 \end{vmatrix} = v_1w_2 - v_2w_1.$$

Note if you wrote out $\mathbf{v}$ in the standard basis ($(1,0)$ and $(0,1)$) it is $\mathbf{v} = (v_1\cos(\theta) - v_2\sin(\theta), v_1\sin(\theta) + v_2\cos(\theta))$ and computing directly that $\|\mathbf{v}\|^2 = v_1^2 + v_2^2$ involves a fair amount of algebra and using that $\sin^2 + \cos^2 = 1$ at least twice. Likewise for the other formulas.

To look at an example of this consider a curve whose equation in polar coordinates is

$$r = \rho(\theta)$$

Using that the x and y of rectangular coordinates are related to the r and θ of polar coordinates by the x and y of rectangular coordinates are related by

$$x = r\cos(\theta) \tag{1}$$

$$y = r\sin(\theta) \tag{2}$$

we have that a parameterization of the curve in rectangular coordinates is

$$\mathbf{c}(\theta) = (\rho(\theta)\cos(\theta), \rho(\theta)\sin(\theta)).$$

This can be written as more compactly as

$$\mathbf{c} = \rho\mathbf{e}_1 \tag{3}$$

(to keep the notation shorter, and easier to read, we are suppressing θ, but keep in mind that $\rho = \rho(\theta)$ and $\mathbf{e}_1 = \mathbf{e}_1(\theta)$ depend on θ.)

Problem 1 (Arclength in polar coordinates). Verify the following: With $\mathbf{c}$ given by Equation (3) show that the velocity vector is

$$\mathbf{v} = \mathbf{c}' = \rho'\mathbf{e}_1 + \rho\mathbf{e}_2.$$

and therefore the speed is

$$v = \|\mathbf{c}'\| = \sqrt{\rho^2 + (\rho')^2}.$$

Therefore the part of the curve with $\alpha \le \theta \le \beta$ has length

$$L = \int_\alpha^\beta \sqrt{\rho(\theta)^2 + \rho'(\theta)^2}\, d\theta.$$ □

Problem 2. This problem is mostly a bit of review of calculus. Let $a > 0$ and consider the curve with polar equation

$$r = 2a\cos(\theta).$$

(a) Show that this is the circle with rectangular equation $(x-a)^2+y^2 = a^2$ and thus this circle has center $(a, 0)$ and radius a.
(b) Show that as θ moving from $-\pi/2$ to $\pi/2$ corresponds to moving around the circle once. (Proof by picture is fine for this.)
(c) Use these facts and the previous problem to show that the length of a circle of radius a is $2\pi a$. (I admit this is not the best way to see this, but it is a good test case for seeing we have the correct formula for the length.) □

Problem 3 (Curvature in polar coordinates). This continues Problem 1. Show that

$$\mathbf{c}'' = (\rho'' - \rho)\mathbf{e}_1 + 2\rho'\mathbf{e}_2$$

and therefore

$$\mathbf{c}' \wedge \mathbf{c}'' = \rho^2 + 2(\rho')^2 - \rho\rho''.$$

Recalling that curvature is given by

$$\kappa = \frac{\mathbf{c}' \wedge \mathbf{c}''}{v^3}$$

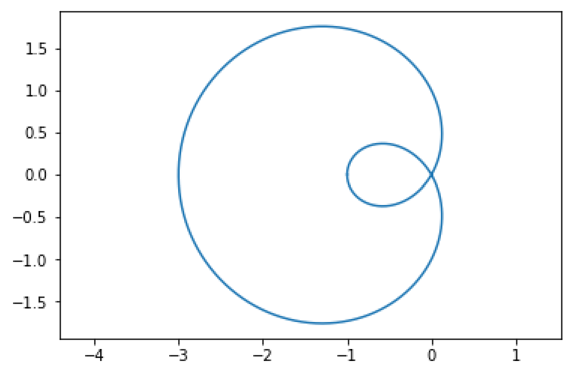
put the pieces together to show that in the case at hand

$$\kappa = \frac{\rho^2 + 2(\rho')^2 - \rho\rho''}{(\rho^2 + (\rho')^2)^{3/2}}. \qquad \square$$

Problem 4. The curve with polar equation

$$r = 1 - 2\cos(\theta)$$

has the following graph:



Compute the curvature, $\kappa(\theta)$, of this curve and show that it has a maximum when $\theta = 0$, a minimum when $\theta = \pi$, and κ has no other local maximums or minimums. $\square$

2. Implicitly defined curves.

Often curves are given by equations such as

$$x^2 + y^2 = 1$$

or

$$\frac{x^2}{a^2} - \frac{y^2}{b} = 1.$$

It is nice to be able find the curvature of such curves without having to find a parameterization.

Let U be an open subset of $\mathbb{R}^2$ and $f\colon U \to \mathbb{R}$ be a function with continuous first and second partial derivatives. Let $\mathbf{c}(x) = (x(t), y(t))$. Then the chain rule is

$$\frac{d}{dt} f(\mathbf{c}(t)) = \nabla f(\mathbf{c}(t)) \cdot \mathbf{c}'(t).$$

Written out in coordinates this is

$$\frac{d}{dt} f(x(t), y(t)) = \frac{\partial f}{\partial x}(x(t), y(t))\frac{dx}{dt} + \frac{\partial f}{\partial y}(x(t), y(t))\frac{dy}{dt}.$$

Better yet is to use subscripts for partial derivative (i.e. $f_x = \frac{\partial f}{\partial x}$, $f_{xy} = \frac{\partial^2 f}{\partial x \partial y}$ etc.) and write this as

$$\frac{dt}{dt} f = f_x x' + f_y y'.$$

If we apply this to $f_x(x(t), y(t))$ we get

$$\frac{d}{dt} f_x = f_{xx} x' + f_{xy} y'.$$

Problem 5. Put these pieces together to get

$$\frac{d^2}{dt^2} f = f_{xx}(x')^2 + 2 f_{xy} x' y' + f_{yy}(y')^2 + f_x x'' + f_y y''. \tag{4}$$

where f_x and the other partial derivative are evaluated at $(x(t), y(t))$

Now assume that $\mathbf{c}(s) = (x(s), y(s))$ is a unit speed curve and

$$f(x(s), y(s)) = C$$

where C is a constant. Taking the derivative of this gives

$$f_x x' + f_y y' = \nabla f(\mathbf{c}(s)) \cdot \mathbf{c}'(t) = 0.$$

That is the tangent vector to the curve, $\mathbf{c}'(s)$, is orthogonal to the gradient $\nabla f = (f_x, f_y)$. We assume that we are moving so that the unit tangent to the curve is

$$(x'(s), y'(s)) = \mathbf{c}'(s) = \mathbf{t}(s) = \frac{1}{\sqrt{f_x^2 + f_y^2}}(-f_y, f_x).$$

Thus

$$x'(s) = \frac{f_x}{\sqrt{f_x^2 + f_y^2}}$$
$$y'(s) = \frac{f_y}{\sqrt{f_x^2 + f_y^2}}$$

Then the unit normal to the curve is

$$\mathbf{n}(s) = \frac{1}{\sqrt{f_x^2 + f_y^2}}(-f_x, -f_y).$$

By the Frenet formulas we have

$$(x''(s), y''(s)) = \mathbf{c}''(s) = \mathbf{t}'(s) = \kappa(s)\mathbf{n}(s) = \frac{1}{\sqrt{f_x^2 + f_y^2}}(-f_x, -f_y).$$

and therefore

$$x'' = \frac{-\kappa f_x}{\sqrt{f_x^2 + f_y^2}}$$
$$y'' = \frac{-\kappa f_y}{\sqrt{f_x^2 + f_y^2}}$$

Problem 6. Use the these formulas in equation (4) to show that the curvature of $\mathbf{c}$ is

$$\kappa = \frac{f_{yy}f_x^2 - 2f_{xy}f_xf_y + f_{xx}f_y^2}{(f_x^2 + f_y^2)^{\frac{3}{2}}}. \qquad \square$$

3. Evolutes and the Tait-Kneser Theorem.

Let $c\colon [a,b] \to \mathbb{R}^2$ be a C^3 curve with curvature κ everywhere positive. Then the radius of curvature for the curve at $\mathbf{c}(s)$ is

$$\rho(s) = \frac{1}{\kappa(s)}.$$

The point

$$\mathbf{E}(s) = \mathbf{c}(s) + \rho(s)\mathbf{n}(s)$$

is the ***center of curvature*** of $\mathbf{c}$ at $\mathbf{c}(s)$ and the ***osculating circle*** at $\mathbf{c}(s)$ is the circle with center $\mathbf{E}(s)$ and radius $\rho(s)$. This is the circle that is tangent to $\mathbf{c}$ at $\mathbf{c}(s)$ and has the same curvature as the curve and therefore is the circle that "best fits" $\mathbf{c}$ at $\mathbf{c}(s)$. The curve $\mathbf{E}\colon [a,b] \to \mathbb{R}^2$ is the ***evolute*** of $\mathbf{c}$.

Problem 7. Use the Frenet formulas to show that

$$\mathbf{E}'(s) = \rho'(s)\mathbf{n}(s).$$

Then use this to show that the unit normal $\mathbf{n}_E(s)$ to $\mathbf{E}$ is

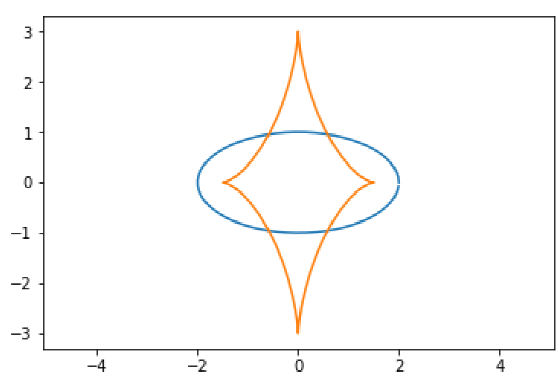
$$\mathbf{t}_E(s) = \begin{cases} \mathbf{n}(s), & \rho'(s) > 0; \\ -\mathbf{n}(s), & \rho'(s) < 0. \end{cases}$$

Use this to show that $\mathbf{n}_E$ flips direction by π radians at any point there ρ has a local maximum or minimum. That is at the vertices of $\mathbf{c}$. $\square$

Here is a picture of the ellipse

$$\frac{x^2}{2^2} + y^2 = 1$$

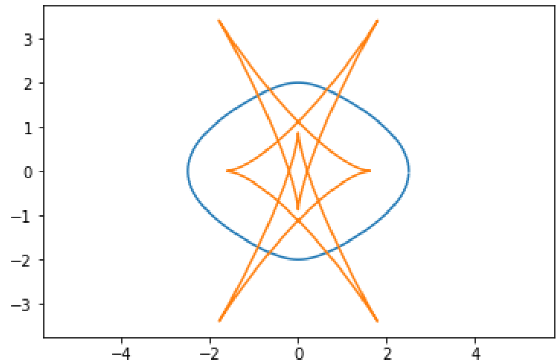
together with its evolute.



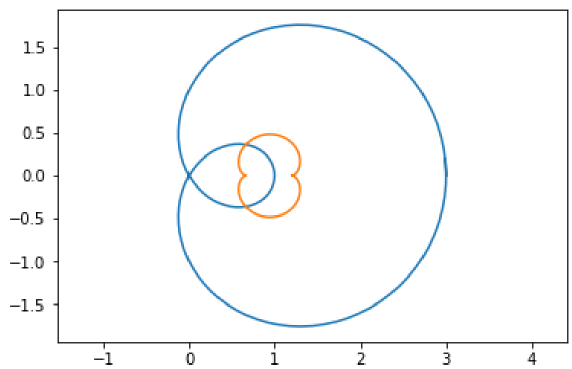
For a more exotic example, or at least one with more vertices, here is the graph of the curve

$$\mathbf{c}(t) = ((2 + .5\cos(2 * t))\cos(t), 2\sin(t)) \qquad 0 \le t \le 2\pi.$$

together with its evolute.



As one last example recall that curve with polar equation $r = 1 + 2\cos(\theta)$ was our example of a closed, but not simple, curve that only has two vertices. Thus its evolute should only have two cusps. Here is the picture showing this is the case:



Problem 8. Let $\mathbf{c}\colon [a, b] \to \mathbb{R}^2$ be a curve where $\kappa > 0$ and is monotone (that is either increasing or decreasing) on the interval. Show that the length of the evolute $\mathbf{E}$ is

$$\text{Length}(\mathbf{E}) = |\rho(b) - \rho(a)|.$$

Hint: The arclength formula is

$$\text{Length}(\mathbf{E}) = \int_a^b \|\mathbf{E}'(s)\|\, ds.$$

Now use that $\mathbf{E}'(s) = \rho'(s)\mathbf{n}$ and that since κ, and therefore also ρ, is monotone that ρ' is either always positive or always negative. $\square$

Problem 9. With the same hypothesis as the previous problem, show

$$\|\mathbf{E}(b) - \mathbf{E}(a)\| < |\rho(b) - \rho(a)|.$$

Hint: For a curve that is not a line segment, its length is greater than the distance between its endpoints. Or put more succinctly, the shortest path between two points is a straight line. □

Problem 10. Let $\mathbf{P}_1$ and $\mathbf{P}_2$ be points in the plane, R_1, R_2 positive real numbers, and let $\mathcal{C}_j$ be the circle with center $\mathbf{P}_j$ and radius R_j. Show that if $\|\mathbf{P}_1 - \mathbf{P}_2\| < |R_1 - R_2|$ then one of the circles $\mathcal{C}_1$ or $\mathcal{C}_2$ is contained in the other one. *Hint:* Proof by picture is fine, and even preferred. □

Theorem 1 (Tait-Kneser Theorem[1]). *Let* $\mathbf{c}\colon [a,b] \to \mathbb{R}^3$ *be a* C^3 *unit speed curve that has positive curvature. Also assume that* κ *is monotone. Then the osculating circles of the curve are nested. That if for any pair of them, one is contained inside the other.*

Problem 11. Prove this. *Hint:* Follow the outline of what we did in class. □

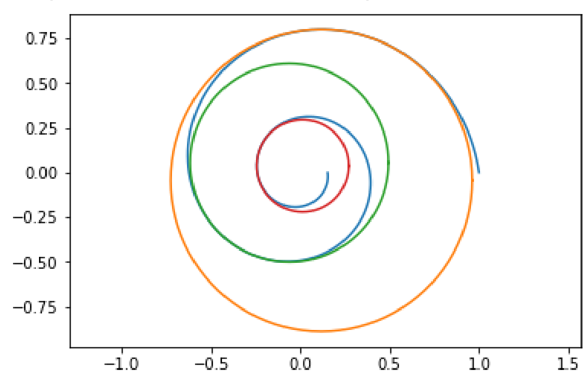
The following figure shows the curve $\mathbf{c}\colon [0, 4\pi] \to \mathbb{R}^2$ given by

$$\mathbf{c}(t) = (e^{-.15t}\cos(t), e^{-.15}\sin(t))$$

(which has polar equation $r = e^{-.15\theta}$) which has curvature

$$\kappa(t) = \frac{20e^{.15t}}{\sqrt{409}}$$

which is increasing. Three of the osculating circles of the curve are shown.



The Tait-Kneser Theorem has some nice consequences. Note that a curve with positive curvature can cross itself many times.

Problem 12. Draw a curve with positive curvature that crosses itself four times. □

Problem 13. Show that a curve $\mathbf{c}\colon [a,b] \to \mathbb{R}^2$ that has positive increasing curvature is embedded (the term ***embedded*** in this context just means that the curve does not cross itself). □

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[1] This result was orginially proven by Peter Tait in a paper published in 1896. Adolf Kneser rediscovered it and published a proof in 1912.