Mathematics 551 Take home part of Test 1

This is due at the beginning of class on Friday, February 14.

We have derived the Frenet formulas for a unit speed curve and it would be a good idea to read the derivation in Shifrin's book:

`https://math.franklin.uga.edu/sites/default/files/users/user317/ShifrinDiffGeo.pdf`

(or see link on the class web page) section 2 pages 10–13. In our notation these are

$$\begin{aligned}
\mathbf{c}' &= \mathbf{t} \\
\mathbf{t}' &= \qquad \kappa\mathbf{n} \\
\mathbf{n}' &= -\kappa\mathbf{t} \qquad + \tau\mathbf{b} \\
\mathbf{b}' &= \qquad -\tau\mathbf{n}
\end{aligned}$$

where κ is the curvature, τ is the torsion, $\mathbf{t}$ is the unit tangent, $\mathbf{n}$ is the unit normal, and $\mathbf{b}$ is the binormal.

In applications to physics engineering and so on, we think of $\mathbf{c}\colon [a,b] \to \mathbb{R}^3$ as $\mathbf{c}(t)$ being the position of a moving point at time t and questions about the motion of the point are as interesting as questions about just the geometry of the curve. So let $\mathbf{c}\colon [a,b] \to \mathbb{R}^3$ have as many derivatives as we need. Think of t as time and let $s = s(t)$ be arclength along $\mathbf{c}$, that is

$$s(t) = \int_a^t \|\mathbf{c}'(u)|\, du$$

so that

$$\frac{ds}{dt} = \|\mathbf{c}'(t)\| = v$$

where v is the speed of the point. Now we rewrite the Frenet formulas in terms of derivatives in terms of t. To start we have

$$\begin{aligned}
\frac{d\mathbf{c}}{ds} &= \mathbf{t} \\
\frac{d\mathbf{t}}{ds} &= \qquad \kappa\mathbf{n} \\
\frac{d\mathbf{n}}{ds} &= -\kappa\mathbf{t} \qquad + \tau\mathbf{b} \\
\frac{d\mathbf{b}}{ds} &= \qquad -\tau\mathbf{n}
\end{aligned}$$

Problem 1. (5 points) Use the chain rule: $\dfrac{d}{dt} = \dfrac{ds}{dt}\dfrac{d}{ds} = v\dfrac{d}{ds}$ to show

$$\begin{aligned} \frac{d\mathbf{c}}{dt} &= v\mathbf{t} \\ \frac{d^2\mathbf{c}}{dt^2} &= \frac{dv}{dt}\mathbf{t} + v^2\kappa\mathbf{n} \\ \frac{d^3\mathbf{c}}{dt^3} &= \left(v\frac{dv}{dt} - v^3\kappa^2\right)\mathbf{t} + \left(v\frac{dv}{dt}\kappa + \frac{d(v^2\kappa)}{ddt}\right)\mathbf{n} + v^3\kappa\tau\mathbf{b}. \end{aligned}$$

□

Problem 2. (5 points) Using the dot notation for time derivatives, that is $\frac{du}{dt} = \dot{u}$, use the formulas of the previous problem so show

$$\kappa = \frac{\|\dot{\mathbf{c}} \times \ddot{\mathbf{c}}\|}{v^3}$$

and

$$\tau = \frac{(\dot{\mathbf{c}} \times \ddot{\mathbf{c}}) \cdot \dddot{\mathbf{c}}}{\|\dot{\mathbf{c}} \times \ddot{\mathbf{c}}\|^2}.$$

In these formulas $\times$ is the vector cross product. □

To use some notation common in science and engineering let

$$\begin{aligned} \mathbf{v} &= \dot{\mathbf{c}} && \text{(the velocity vector)} \\ \mathbf{a} = \dot{\mathbf{v}} &= \ddot{\mathbf{c}} && \text{(the acceleration vector).} \end{aligned}$$

With this notation Newton's second law (force is mass times acceleration):

$$\mathbf{F} = m\mathbf{a}$$

becomes

$$\mathbf{F} = m\ddot{\mathbf{c}}$$

In the next problem we combine this with the Frenet to get a basic result in particle physics. It is also yet anther good example of the technique of getting new results by taking repeated derivatives of given formulas.

Problem 3 (Path of a charged particle in a magnetic field). (15 points) A standard model for the force on a particle moving in a magnetic field is

$$\mathbf{F} = q\mathbf{v} \times \mathbf{B}$$

where $\mathbf{v}$ is the vector of the particle and $\mathbf{B}$ is is the magnetic field and q is the charge of the particle. We will assume that $\mathbf{B}$ is constant. Then Newton's second law gives if $\mathbf{c}(t)$ is the position of the particle at time t that

$$m\frac{d^2\mathbf{c}}{dt^2} = q\frac{d\mathbf{c}}{dt} \times \mathbf{B}. \tag{1}$$

where m is the mass of the particle and q is its charge. Here we will show this implies the particle moves on a Helix. To do this it is enough to show the curvature and torsion are constant.

(a) Show the speed of $\mathbf{c}$ is constant. *Hint:* it is enough to show

$$\frac{d}{dt}\left\|\frac{d\mathbf{c}}{dt}\right\|^2 = 0.$$

To see this use that (1) and a basic property of cross products implies that $\dot{\mathbf{c}}$ and $\ddot{\mathbf{c}}$ are orthogonal. For the rest of this problem we let

$$v_0 = \left\|\frac{d\mathbf{c}}{dt}\right\|$$

be the speed of the particle.

(b) Let s be the arclength be arclength along $\mathbf{c}$. Show

$$\frac{d\mathbf{c}}{dt} = v_0\frac{d\mathbf{c}}{ds}, \qquad \frac{d^2\mathbf{c}}{dt^2} = v_0^2\frac{d^2\mathbf{c}}{ds^2}. \tag{2}$$

(c) Use parts (a) and (b) of this problem along with Newton's second law to show

$$\frac{d^2\mathbf{c}}{ds^2} = \frac{d\mathbf{c}}{ds}\times\mathbf{A} \tag{3}$$

where $\mathbf{A}$ is the constant vector

$$\mathbf{A} = \left(\frac{q}{nv_0}\right)\mathbf{B}.$$

(d) Use (3) to show

$$\frac{d^3\mathbf{c}}{ds^3} = \frac{d^2\mathbf{c}}{ds^2}\times\mathbf{A}, \qquad \frac{d^4\mathbf{c}}{ds^4} = \frac{d^3\mathbf{c}}{ds^3}\times\mathbf{A}$$

and therefore

$$\left\|\frac{d^2\mathbf{c}}{ds^2}\right\| \quad \text{and} \quad \left\|\frac{d^3\mathbf{c}}{ds^3}\right\|$$

are constant.

(e) Let θ be the angle between $\frac{d\mathbf{c}}{ds}$ and $\mathbf{A}$ (which is the same as the angle between $\frac{d\mathbf{c}}{dt}$ and $\mathbf{B}$). Show θ, the curvature κ, and the torsion τ are all constant. (This shows the motion of the particle is a helix (or a circle if $\tau = 0$) and the axis of the helix is parallel to the direction of $\mathbf{B}$.)

(f) (Optional open ended question.) In a cloud chamber contained inside a constant magnetic field $\mathbf{B}$, what can be observed of a charged particle moving through the chamber is the path of the particle. That is its axis (which we know to be parallel to $\mathbf{B}$), the curvature, the torsion, and θ, the angle the tangent to the helix makes with $\mathbf{B}$. Given this information how much can be deduced about the charge, q, the mass, m, and the speed, v_0 of the particle? □

Problem 4. (15 points) In this problem and the next you will answer the question: What are the conditions on the curvature and torsion that imply a curve is a subset of a sphere? To start let $\mathbf{c}\colon [a,b] \to \mathbb{R}^3$ be be a unit speed curve that is on the sphere with center $\mathbf{E}$ and radius R. We also assume that κ and τ never vanish. Then for $t \in [a,b]$ we have

$$\|\mathbf{c}(t) - \mathbf{E}\|^2 = R^2.$$

As usual we take a derivative. Using the $\mathbf{E}$ and R are constant and using $\mathbf{c}'(t) = \mathbf{t}(t)$

$$2\mathbf{t}(t) \cdot (\mathbf{c}(t) - \mathbf{E}) = 2\mathbf{c}'(t) \cdot (\mathbf{c}(t) - \mathbf{E}) = 0,$$

Thus $\mathbf{c}(t) - \mathbf{E}$ is orthogonal to $\mathbf{T}$. Therefore $\mathbf{c}(t) - \mathbf{E}$ is a linear combination of $\mathbf{n}(t)$ and $\mathbf{b}(t)$:

$$\mathbf{c} - \mathbf{E} = u\mathbf{n} + v\mathbf{b}$$

for functions $u, v \colon [a,b] \to \mathbb{R}$. This can be rewritten as

$$\mathbf{E} = \mathbf{c} + u\mathbf{n} + v\mathbf{b}. \tag{4}$$

(a) Take the derivative of (4) and use that $\mathbf{E}$ is constant and the Frenet formulas to get the equation

$$\mathbf{0} = (1 - u\kappa)\mathbf{t} + (u' - v\tau)\mathbf{n} + (u\tau + v')\mathbf{b}$$

and thus

$$1 - u\kappa = 0, \qquad u' - v\tau = 0 \qquad u\tau + v' = 0$$

(b) Show these imply

$$u = \frac{1}{\kappa}$$

$$v = \frac{1}{\tau}\left(\frac{1}{\kappa}\right)'$$

$$\frac{\tau}{\kappa} + \left(\frac{1}{\tau}\left(\frac{1}{\kappa}\right)'\right)' = 0$$

(c) Conclude that if $\mathbf{c}$ is on a sphere that

$$\frac{\tau}{\kappa} + \left(\frac{1}{\tau}\left(\frac{1}{\kappa}\right)'\right)' = 0$$

holds along the curve. □

Problem 5. (15 points) Let $\mathbf{c} \colon [a,b] \to \mathbb{R}^3$ be any curve with nonvanishing curvature and torsion and set

$$\mathbf{E} = \mathbf{c} + \frac{1}{\kappa}\mathbf{n} + \frac{1}{\tau}\left(\frac{1}{\kappa}\right)'\mathbf{b}$$

(a) Show

$$\mathbf{E}' = \left(\frac{\tau}{\kappa} + \left(\frac{1}{\tau}\left(\frac{1}{\kappa}\right)'\right)'\right)\mathbf{b}$$

Hint: This maybe an bit more transparent if you let

$$\mathbf{E} = \mathbf{c} + u\mathbf{n} + v\mathbf{b}$$

with

$$u = \frac{1}{\kappa} \quad \text{and} \quad v = \frac{1}{\tau}\left(\frac{1}{\kappa}\right)'.$$

Then you can just refer to a calculation done in Problem 4 to get the result.

(b) Conclude that $\mathbf{E}(t)$ is constant if and only if

$$\frac{\tau}{\kappa} + \left(\frac{1}{\tau}\left(\frac{1}{\kappa}\right)'\right)' = 0. \tag{5}$$

(c) Show that if (5) holds on $\mathbf{c}$ that

$$\frac{d}{ds}\|\mathbf{c}(s) - \mathbf{E}\|^2 = 0.$$

(d) Finish by explaining why if (5) holds on $\mathbf{c}$, then $\mathbf{c}$ moves on a sphere.

(e) (Optional open ended question.) Is there an anologue of the Tait-Kneser theorem for space curves? In particular let, as above, let

$$\mathbf{E} = \mathbf{c} + \frac{1}{\kappa}\mathbf{n} + \frac{1}{\tau}\left(\frac{1}{\kappa}\right)'\mathbf{b}$$

and

$$\rho = \|\mathbf{E} - \mathbf{c}\| = \sqrt{\left(\frac{1}{\kappa}\right)^2 + \left(\frac{1}{\tau}\left(\frac{1}{\kappa}\right)'\right)^2}.$$

Is there a natural condition on κ and τ that implies the spheres with centers $\mathbf{E}(s)$ and radius $\rho(s)$ are nested?