# Mathematics 551 Homework, February 27, 2024

Contents



## 1. The first fundamental form

Here we will look at parameterizations of some surfaces and do our best to get a feel for the meaning of the first fundamental form.

To start we let $U$ be an open set in $\mathbb{R}^2$ and $\boldsymbol{x}\colon U \to \mathbb{R}^3$ a regular parameterization. That is $\boldsymbol{x}$ is injective and

$$\boldsymbol{x}_u \times \boldsymbol{x}_v \neq \mathbf{0}.$$

Set

$$E := \boldsymbol{x}_u \cdot \boldsymbol{x}_u, \qquad F := \boldsymbol{x}_u \cdot \boldsymbol{x}_v, \qquad G = \boldsymbol{x}_v \cdot \boldsymbol{x}_v.$$

Then the first ***first fundamental form*** of $\boldsymbol{x}$, first defined by Gauss, is

$$I = E\, du^2 + 2F\, du dv + G\, dv^2.$$

Let $\boldsymbol{c}(t) = (u(t), v(t))$ with $a \leq t \leq b$ be a curve in the domain of the parameters and let

$$\boldsymbol{\gamma}(t) = \boldsymbol{x}(\boldsymbol{c}(t)) = \boldsymbol{x}(u(t), v(t))$$

be the image of $\boldsymbol{c}$ under $\boldsymbol{x}$. Then by the chain rule

$$\boldsymbol{\gamma}'(t) = \frac{d}{dt}\boldsymbol{x}(u(t), v(t)) = \boldsymbol{x}_u \dot{u} + \boldsymbol{x}_v \dot{v}.$$

Therefore

$$\|\boldsymbol{\gamma}'(t)\|^2 = \|\boldsymbol{x}_u \dot{u} + \boldsymbol{x}_v \dot{v}\|^2 = E\dot{u}^2 + 2F\dot{u}\dot{v} + G\dot{v}^2.$$

and thus the length of $\boldsymbol{\gamma}$ is

$$L(\boldsymbol{\gamma}) = \int_a^b \sqrt{E\dot{u}^2 + 2F\dot{u}\dot{v} + G\dot{v}^2}\, dt$$

It we use the usual notation for differentials $du = \dot{u}\,dt$ and $dv = \dot{v}\,dt$ then $dt = \sqrt{dt^2}$ can be moved under the square root to give

$$L(\boldsymbol{\gamma}) = \int_a^b \sqrt{E\,du^2 + 2F\,dudv + G\,dv^2}.$$

Thus arc length is found by integrating $\sqrt{E\,du^2 + 2F\,dudv + G\,dv^2}$ along the curve. So classically (in the days of Euler, Lagrange, Gauss, etc.) this would be written as

$$ds = \sqrt{E\,du^2 + 2F\,dudv + G\,dv^2}$$

and so anther notation for the first fundamental is

$$ds^2 = I = E\,du^2 + 2F\,dudv + G\,dv^2.$$

Here is a more modern point of view. Let $M$ denote the image of $\boldsymbol{x}$ and let $(u_0, v_0) \in U$ be a point in the domain of the domain, $U$, of the parameters and let $P = \boldsymbol{x}(u_0, v_0)$ be the corresponding point on $M$. We would like to define the tangent space to $M$ at $P$. Let $\boldsymbol{c}\colon (-\delta, \delta) \to U$ with $\boldsymbol{c}(0) = (u_0, v_0)$ be a curve in $U$ through $(u_0, v_0)$. Write this in the coordinates on $U$ as

$$\boldsymbol{c}(t) = (u(t), v(t)$$

Then the curve

$$\boldsymbol{\gamma}(t) = \boldsymbol{x}(\boldsymbol{c}(t)) = \boldsymbol{x}(u(t), v(t))$$

is a curve in $M$ with

$$\boldsymbol{\gamma}(0) = \boldsymbol{x}(u(0), v(0)) = \boldsymbol{x}(u_0, v_0) = P.$$

Thus $\boldsymbol{\gamma}$ is a curve in the surface with through $P$. Our definition of the ***tangent space*** to $M$ at $P$, denoted $T_P M$, is the set of all tangent vectors to such curves. Using the chain rule we have

$$\boldsymbol{\gamma}'(t) = \frac{d}{dt}\boldsymbol{x}(u(t), v(t))\bigg|_{t=0} = u'(0)\boldsymbol{x}_u(u_0, v_0) + v'(0)\boldsymbol{x}(u_0, v_0)$$

which is a linear combination of the vectors $\boldsymbol{x}_u(u_0, v_0)$ and $\boldsymbol{x}(u_0, v_0)$. So an equivalent definition of $T_P M$, and the one we will work with, is

$$T_P M = \{\xi \boldsymbol{x}_u(u_0, v_0) + \eta \boldsymbol{x}_v(u_0, v_0) : \xi, \eta \in \mathbb{R}\}$$

which is just set theoretic notation for the set of all linear combinations of $\boldsymbol{x}_u(u_0, v_0)$ and $\boldsymbol{x}_v(u_0, v_0)$. We now relate the first fundamental form, $I$, to the inner product of tangent vectors in the tangent space. Let $\boldsymbol{v}_1 = \xi_1 x_u(u_0, v_0) + \eta_1 \boldsymbol{x}_v(u_0, v_0)$ and $\boldsymbol{v}_2 = \xi_2 \boldsymbol{x}_u(u_0, v_0) + \eta_2 \boldsymbol{x}_v(u_0, v_0)$. Then the inner product of these vectors is

$$\begin{aligned}\boldsymbol{v}_1 \cdot \boldsymbol{v}_2 &= (\xi_1 x_u(u_0, v_0) + \eta_1 \boldsymbol{x}_v(u_0, v_0)) \cdot (\xi_2 \boldsymbol{x}_u(u_0, v_0) + \eta_2 \boldsymbol{x}_v(u_0, v_0)) \\ &= E(u_0, v_0)\xi_1\xi_2 + F(u_0, v_0)(\xi_1\eta_1 + \xi_2\eta_1) + G(u_0, v_0)\eta_1\eta_2.\end{aligned}$$

Or in somewhat less precise, but more readable notation,

$$\boldsymbol{v}_1 \cdot \boldsymbol{v}_2 = E\xi_1\xi_2 + F(\xi_1\eta_1 + \xi_2\eta_1) + G\eta_1\eta_2. \tag{1}$$

With this notation we can also view $I_P$ as a function, $I_P(\boldsymbol{v}_1, \boldsymbol{v}_2)$, of pairs of vectors $v_1, \boldsymbol{v}_2 \in T_P M$ by the formula (1).

**Proposition 1.** *With this definition $I_P$ is a symmetric form on the tangent space $T_P M$. That is*

*(a) $I_P$ is bilinear, that is it is a linear function of each of its arguments: For any scalars $c$ and $c'$*

$$\begin{aligned} I_P(c\boldsymbol{v}_1 + c'\boldsymbol{v}_1', \boldsymbol{v}_2) &= cI_P(\boldsymbol{v}_1, \boldsymbol{v}_2) + c' I_P(\boldsymbol{v}_1', \boldsymbol{v}_2) \\ I_P(\boldsymbol{v}_1, c\boldsymbol{v}_2 + c'\boldsymbol{v}_2') &= cI_P(\boldsymbol{v}_1, \boldsymbol{v}_2) + c' I_P(\boldsymbol{v}_1, \boldsymbol{v}_2') \end{aligned}$$

*(b) $I_P$ is symmetric,*

$$I_P(\boldsymbol{v}_1, \boldsymbol{v}_2) = I_P(\boldsymbol{v}_2, \boldsymbol{v}_1).$$

**Problem** 1. Write out enough of a proof of this so that you believe it. Do not hand this in. □

Here is an example. Assume a regular parameterization, $\boldsymbol{x}$ with first fundamental form

$$I = (1 + u^2) du^2 + 2uv\, dudv + (1 + v^2)\, dv^2.$$

for some surface $M$. Let

$$\boldsymbol{a} = 2\boldsymbol{x}_u(1, 2) + 3\boldsymbol{x}_v(1, 2), \qquad \boldsymbol{b} = 4\boldsymbol{x}_u(1, 2) + 5\boldsymbol{x}(1, 2)$$

be vectors tangent to $M$ at $P = \boldsymbol{x}(1, 2)$. Let us find the length of these vectors and the angle between them. The first fundamental form of $M$ at $P$ is

$$I_P = (1 + 1^2) du^2 + 2(1)(2) + (1 + 2^2) dv^2 = 2du^2 + 4dudv + 5dv^2.$$

This implies that at $P$

$$I_P(\boldsymbol{x}_u, \boldsymbol{x}_u) = 2, \qquad I_P(\boldsymbol{x}_u, \boldsymbol{x}_v) = 2, \qquad I_P(\boldsymbol{x}_v, \boldsymbol{x}_v) = 5.$$

So using the bilinearity and symmetry of $I_P$ we get at $P$ that

$$\begin{aligned} I_P(\boldsymbol{a}, \boldsymbol{a}) &= I_P(2\boldsymbol{x}_u + 3\boldsymbol{x}_v, 2\boldsymbol{x}_u + 3\boldsymbol{x}_v) \\ &= 2^2 I_P(\boldsymbol{x}_u, \boldsymbol{x}_u) + 2(2)(3) I_P(\boldsymbol{x}_u, \boldsymbol{x}_v) + 3^2 I_P(\boldsymbol{x}_u, \boldsymbol{x}_v) \\ &= 4(2) + 12(2) + 9(5) \\ &= 77 \\ I_P(\boldsymbol{a}, \boldsymbol{b}) &= I_P(2\boldsymbol{x} + 3\boldsymbol{x}_v, 4\boldsymbol{x}_u + 5\boldsymbol{x}_v) \\ &= 2^2 I_P(\boldsymbol{x}_u, \boldsymbol{x}_u) + ((2)(5) + (3)(4)) I_P(\boldsymbol{x}_u, \boldsymbol{x}_v) + (3)(5) I_P(\boldsymbol{x}_u, \boldsymbol{x}_v) \\ &= 4(2) + 22(2) + 15(5) \\ &= 127 \\ I_P(\boldsymbol{b}, \boldsymbol{b}) &= I_P(4\boldsymbol{x} + 5\boldsymbol{x}_v, 4\boldsymbol{x}_u + 5\boldsymbol{x}_v) \\ &= 4^2 I_P(\boldsymbol{x}_u, \boldsymbol{x}_u) + 2(4)(5) I_P(\boldsymbol{x}_u, \boldsymbol{x}_v) + 5^2 I_P(\boldsymbol{x}_u, \boldsymbol{x}_v) \\ &= 16(2) + 40(2) + 25(5) \\ &= 237 \end{aligned}$$

Therefore

$$\|\boldsymbol{a}\| = \sqrt{77}$$
$$\|\boldsymbol{b}\| = \sqrt{237}$$

and it $\theta$ is the angle between $\boldsymbol{a}$ and $\boldsymbol{b}$

$$\boldsymbol{a}\cdot\boldsymbol{b} = I_P(\boldsymbol{a},\boldsymbol{b}) = 127 = \|1a\|\|\boldsymbol{b}\|\cos(\theta) = \sqrt{67}\sqrt{237}\cos(\theta).$$

Therefore

$$\theta = \arccos\left(127/\sqrt{(77)(237)}\right) = \arccos\left(127/\sqrt{18{,}249}\,\right)$$

**Problem** 2. Let $\boldsymbol{x}$ be a parameterization of a surface $M$ with first fundamental

$$ds^2 = (1+u^2v^2)du^2 + 2uv\,dudv + (2+u^2v^2)\,dv^2.$$

Let $P = \boldsymbol{x}(1,0)$ and let

$$\boldsymbol{a} = 2\boldsymbol{x}_u(1,0) - 3\boldsymbol{x}_v(1,0), \qquad \boldsymbol{b} = -\boldsymbol{x}_u(1,0) + 4\boldsymbol{x}(1,0).$$

Find the length of $\boldsymbol{a}$ and $\boldsymbol{b}$ and the angle between $\boldsymbol{a}$ and $\boldsymbol{b}$. □

**Problem** 3. Let $I = E\,du^2 + 2F\,dudv + G\,dv^2$ be the first fundamental form of a parameterization $\boldsymbol{x}$. Show that it $\theta$ is the angle between $\boldsymbol{x}_u$ and $\boldsymbol{x}_v$, then

$$\cos(\theta) = \frac{\sqrt{E}\sqrt{G}}{F}.$$

Use that $|\cos(\theta)| < 1$ to conclude that $F^2 < EG$. (We have that $|\cos(t)| < 1$ because if $|\cos(\theta)| = 1$, then $\theta = 0$ (when $\cos(\theta) = 1$) or $\theta = \pi$ (when $\cos(\theta) = -1$. But in these two cases ether $\boldsymbol{x}_u$ and $\boldsymbol{x}_v$ either point in the same direction or in opposite directions, contradicting that $\boldsymbol{x}_i$ and $\boldsymbol{x}_v$ are linearly independent). □

## 2. Examples of parameterizations of surfaces

2.1. **Monge patches.** . Here is what is the most basic example. Let $U \subseteq \mathbb{R}^2$ be an open set and $f\colon U \to \mathbb{R}$. Then

$$\boldsymbol{x}(u,v) = (u, v, f(u,v))$$

for $(u,v) \in U$ parameterizes the graph of $f$. Such a parameterization is called a ***Monge patch***.

**Problem** 4. Show the first fundamental form of a Monge patch is

$$ds^2 = (1+f_u^2)\,du^2 + 2f_uf_v\,dydv + (1+f_v^2)\,dv^2$$

□

For an example of a Monge patch see Figure 1.

Even if a surface is not originally thought of as a graph, it can still be useful to parameterize parts of it with a Monge patch. See figure 2

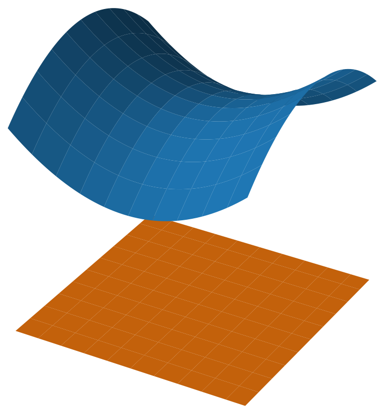

FIGURE 1. The Monge patch for the graph of $z = x^2 - y^2 + 4$ over the domain $U = (-1, 1) \times (-1, 1)$.

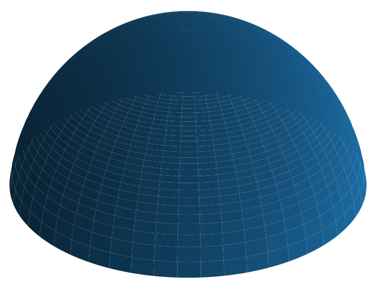

FIGURE 2. The upper half of the sphere $x^2 + y^2 + z^1 = 1$ represented as the graph of $z = \sqrt{1 - x^2 - y^2}$.

2.2. **Cylinders.** Let $\boldsymbol{c}(s) = (x(t), y(t))$ with $a \leq s \leq b$ be curve in the plane. Then the ***cylinder*** over this curve the surface parameterized by

$$\boldsymbol{x}(u, v) = (x(u), y(u), v).$$

This the union of the set of all lines parallel to the $z$-axis and intersect $\boldsymbol{c}$. For an example of a cylinder see Figure

**Problem** 5. Use that $\boldsymbol{c}$ is unit speed to show the first fundamental form of $\boldsymbol{x}$ is

$$ds^2 = (x'(u)^2 + y'(u)^2)\, du^2 + dv^2.$$

Thus if $\boldsymbol{c}$ is unit speed this becomes

$$ds^2 = du^2 + dv^2.$$

2.3. **Rotating frames.** Now let use look at examples of parameterizations of some surfaces. Several calculations are easier if expressed in terms of a

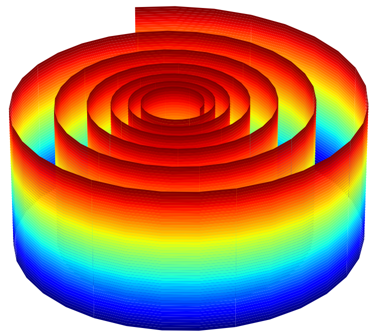

FIGURE 3. A part of the cylinder over the curve $\boldsymbol{c}(t) = e^{t/20}\cos(t), e^{t/20}\sin(t))$.

rotating orthonormal basis. For $\theta \in \mathbb{R}$ define

$$\begin{aligned}\boldsymbol{e}_1(\theta) &= (\cos(\theta), \sin(\theta), 0)\\ \boldsymbol{e}_2(\theta) &= (-\sin(\theta), \sin(\theta), 0)\\ \boldsymbol{e}_3 &= (0, 0, 1).\end{aligned}$$

Formulas that will come up several times are

$$\begin{aligned}\boldsymbol{e}_1'(\theta) &= \boldsymbol{e}_2(\theta)\\ \boldsymbol{e}_2'(\theta) &= -\boldsymbol{e}_1(\theta)\\ \boldsymbol{e}_3' &= 0.\end{aligned}$$

2.4. **Helicoids.** These have parameterization

$$\boldsymbol{x}(u, v) = v\boldsymbol{e}_1(u) + bu\boldsymbol{e}_3$$

where $b$ is a constant.

**Problem** 6. Compute the first fundamental form of the helicoid □

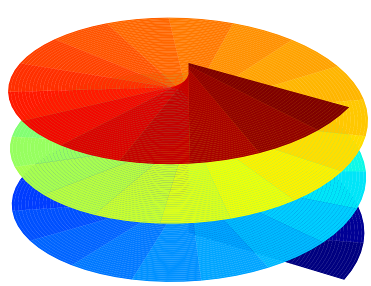

FIGURE 4. A part of the helicoid $\boldsymbol{x}(u, v) = (v\cos(u), v\sin(u), u)$.

2.5. **Surfaces of revolution.** Let $U = \{(x, y) : u > 0\}$ be the right half plane in the $x$-$y$ plane and let. Let

$$\boldsymbol{c}(t) = (x(t), y(t))$$

be a curve in $U$ (so that $x(t) > 0$. Then the surface we get by rotating revolving this curve around the $y$ axis is parameterized

$$\boldsymbol{x}(t, \theta) = x(t)\boldsymbol{e}_1(\theta) + y(t)\boldsymbol{e}_3$$

where we have taken a break from using $u$ and $v$ as the parameter names. Examples of surfaces of revolution are in Figures 5 and 6

**Problem** 7. Compute the first fundamental form of $\boldsymbol{x}$. □

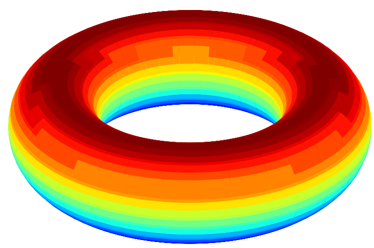

FIGURE 5. The torus formed by revolving the circle $(x - 3)^2 + y^2 = 1$ about the $y$-axis.

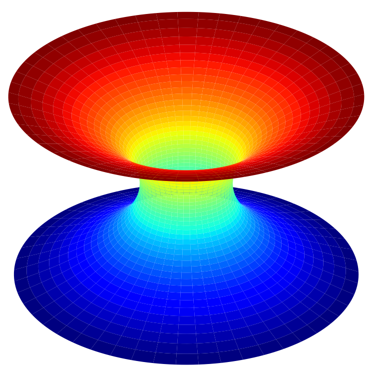

FIGURE 6. Part of the catenoid formed by revolving $x = \cosh(y)$ around the $y$ axis.

**Problem** 8. Figure 7 is the cone $z^2 = x^2 + y^2$. Find a parameterization of the upper half of this cone and compute its first fundamental form. □

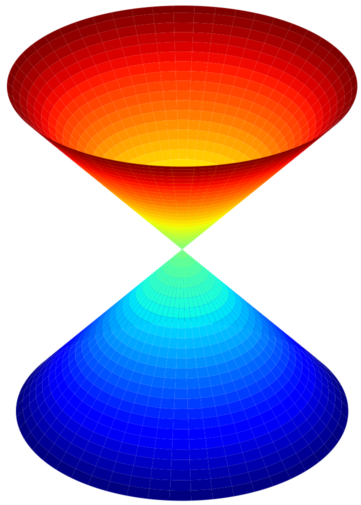

FIGURE 7. Part of the cone defined by $z^2 = x^2 + y^2$.

2.6. **Möbius strip.** A Möbius strip can be parameterized by

$$\boldsymbol{x}(t,\theta) = (2 + t\cos(\theta/2))\boldsymbol{e}_1(\theta) + t\sin(\theta/2)\boldsymbol{e}_3.$$

See Figure 8

**Problem** 9. Compute the first fundamental of this Möbius strip.

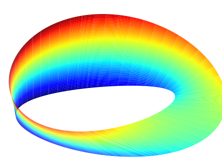

FIGURE 8. A Möbius strip.

2.7. **Tubes around curves.** Let $\boldsymbol{c}\colon [a,b] \to \mathbb{R}^3$ be a unit speed curve and let $r > 0$. Then the ***tube of radius*** $r$ about $\boldsymbol{c}$ is the curve parameterized by

$$\boldsymbol{x}(s,t) = \boldsymbol{c}(s) + r\cos(t)\boldsymbol{n}(s) + r\sin(t)\boldsymbol{b}(s).$$

**Problem** 10. As a refresher about using the Frenet formulas, compute the first fundamental form of this tube. □