Mathematics 551 Homework.

Contents

1. The second fundamental form.

If $\boldsymbol{x}\colon U \to \mathbb{R}^3$ is a local parameterization of a surface, then the unit normal is

$$\boldsymbol{n} = \frac{\boldsymbol{x}_u \times \boldsymbol{x}_v}{\|\boldsymbol{x}_u \times \boldsymbol{x}_v\|}.$$

Then the ***second fundamental form*** is the quadratic form

$$I\!I = L\,du^2 + 2M\,dudv + N\,dv^2$$

where

$$L = \boldsymbol{x}_{uu}\cdot\boldsymbol{n}, \quad M = \boldsymbol{x}_{uv}\cdot\boldsymbol{n}, \qquad \boldsymbol{x}_{vv}\cdot\boldsymbol{n}.$$

In this homework, you compute the normal, $\boldsymbol{n}$ the second fundamental form for some of the examples we look at in a previous example.

2. Examples

2.1. **Monge patches.** . Here is what is the most basic example. Let $U \subseteq \mathbb{R}^2$ be an open set and $f\colon U \to \mathbb{R}$. Then

$$\boldsymbol{x}(u,v) = (u, v, f(u,v))$$

for $(u,v) \in U$ parameterizes the graph of f. Such a parameterization is called a ***Monge patch***.

Problem 1. Find the unit normal and the second fundamental form for a Monge patch. □

Problem 2. Assume that $f(0,0) = f_u(0,0) = f_v(0,0)$ and $f_{uu}(0,0) = a$, $f_{u,v}(0,0) = 0$, and $f_{vv}(0,0) = b$. What is $I\!I$ at the origin? □

2.2. **Cylinders.** Let $\boldsymbol{c}(s) = (x(t), y(t))$ with $a \le s \le b$ be curve in the plane. Then the ***cylinder*** over this curve the surface parameterized by

$$\boldsymbol{x}(u,v) = (x(u), y(u), v).$$

This the union of the set of all lines parallel to the z-axis and intersect $\boldsymbol{c}$. We have seen the first fundamental form is

$$I = (x'(u)^2 + y'(u)^2)\, du^2 + dv^2.$$

which reduces to

$$I = du^2 + dv^2.$$

when the curve is unit speed.

Problem 3. Assume the curve is unit speed and find $I\!I$. Your answer should involve the curvature of $\boldsymbol{c}$. □

2.3. **Rotating frames.** To save you having to look at the old homework, here are the formulas rotating frames. For $\theta \in \mathbb{R}$ define

$$\begin{aligned}\boldsymbol{e}_1(\theta) &= (\cos(\theta), \sin(\theta), 0)\\ \boldsymbol{e}_2(\theta) &= (-\sin(\theta), \sin(\theta), 0)\\ \boldsymbol{e}_3 &= (0,0,1).\end{aligned}$$

Formulas that will come up several times are

$$\begin{aligned}\boldsymbol{e}_1'(\theta) &= \boldsymbol{e}_2(\theta)\\ \boldsymbol{e}_2'(\theta) &= -\boldsymbol{e}_1(\theta)\\ \boldsymbol{e}_3' &= 0.\end{aligned}$$

2.4. **Helicoids.** These have parameterization

$$\boldsymbol{x}(u,v) = v\boldsymbol{e}_1(u) + bu\boldsymbol{e}_3$$

where b is a constant.

Problem 4. Compute the second fundamental form of the helicoid □

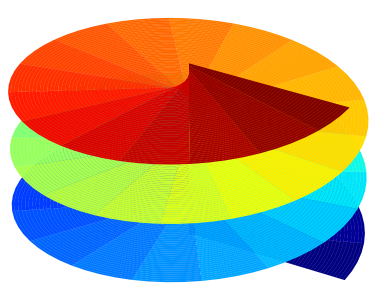

FIGURE 1. A part of the helicoid $\boldsymbol{x}(u,v) = (v\cos(u), v\sin(u), u)$.

2.5. **Surfaces of revolution.** Let $U = \{(x, y) : u > 0\}$ be the right half plane in the x-y plane and let. Let

$$\boldsymbol{c}(t) = (x(t), y(t))$$

be a curve in U (so that $x(t) > 0$. Then the surface we get by rotating revolving this curve around the y axis is parameterized

$$\boldsymbol{x}(t, \theta) = x(t)\boldsymbol{e}_1(\theta) + y(t)\boldsymbol{e}_3$$

where we have taken a break from using u and v as the parameter names. Examples of surfaces of revolution are in Figures 2 and 3

Problem 5. Assume the curve $\boldsymbol{c}$ is unit speed. Compute the second fundamental form of $\boldsymbol{x}$. Your answer should involve the curvature of $\boldsymbol{c}$. □

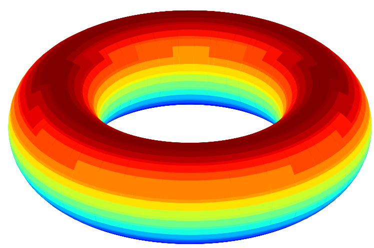

FIGURE 2. The torus formed by revolving the circle $(x - 3)^2 + y^2 = 1$ about the y-axis.

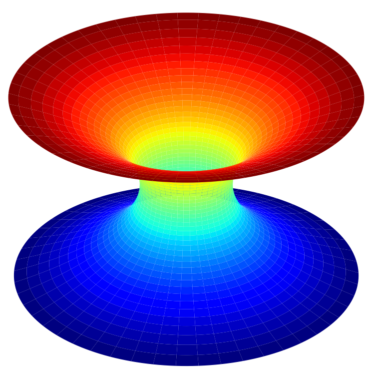

FIGURE 3. Part of the catenoid formed by revolving $x = \cosh(y)$ around the y axis.

2.6. **Tubes around curves.** Let $\boldsymbol{c}\colon [a, b] \to \mathbb{R}^3$ be a unit speed curve and let $r > 0$. Then the ***tube of radius*** r about $\boldsymbol{c}$ is the curve parameterized by

$$\boldsymbol{x}(s, t) = \boldsymbol{c}(s) + r\cos(t)\boldsymbol{n}(s) + r\sin(t)\boldsymbol{b}(s).$$

Problem 6. Find $I\!I$ for this $\boldsymbol{x}$. □