Mathematics 551 Homework.

One or our basic formulas is that for vector fields $\boldsymbol{V}, \boldsymbol{W}$ on a surface M that

$$D_{\boldsymbol{V}}\boldsymbol{W} = \nabla_{\boldsymbol{V}}\boldsymbol{W} + I\!I(\boldsymbol{V}, \boldsymbol{W})\boldsymbol{n}k$$

where $\nabla_{\boldsymbol{V}}\boldsymbol{W}$ is the component of $D_{\boldsymbol{V}}\boldsymbol{W}$ tangent to M. In terms of a local parameterization, $\boldsymbol{x}(u, v)$ this is

$$D_{\boldsymbol{x}_u}\boldsymbol{x}_v = \frac{\partial}{\partial u}\boldsymbol{x}_v = \nabla_{\boldsymbol{x}_u}\boldsymbol{x}_v + I\!I(\boldsymbol{x}_u, \boldsymbol{x}_v)\boldsymbol{n}.$$

If $\boldsymbol{c}(s) \in M$ is a curve in M, then this specializes to

$$\boldsymbol{c}''(s) = \nabla_{\boldsymbol{c}'(s)} + I\!I(\boldsymbol{c}'(s), \boldsymbol{c}'(s))\boldsymbol{n}.$$

Assume that $\boldsymbol{c}$ is unit speed and that its curvature as a space curve is κ, so that by the Frenet formulas we have

$$\boldsymbol{c}''(s) = \kappa(s)\boldsymbol{N}$$

where $\boldsymbol{N}$ is the principle normal to $\boldsymbol{c}$. Note that as $\nabla_{\boldsymbol{c}'(s)}\boldsymbol{c}'(s) \in T_{\boldsymbol{c}(s)}M$ that $\nabla_{\boldsymbol{c}'(s)}\boldsymbol{c}'(s) \cdot \boldsymbol{n} = 0$.

Theorem 1. *With this notation*

$$I\!I(\boldsymbol{c}'(s), \boldsymbol{c}'(s)) = \kappa(s)\boldsymbol{n} \cdot \boldsymbol{N} = \kappa(s)\cos(\theta)$$

where θ the angle between $\boldsymbol{n}$ and $\boldsymbol{N}$. Thus

A ***ruled surface***, M, is a surface such that for each $p \in M$ there is a line of $\mathbb{R}^3$ through p and contained in M. The lines are in M are called the ***rulings*** of M. For pictures of models of ruled surfaces made with wire look here. For more pictures and see the artcle Ruled surfaces and developable surfaces where Johannes Wallner discusses applications (mostly to architecture) of ruled surfaces.

The most obvious (and least interesting) examples are the planes.

Anther set of examples are the cylinders. The official definition of a ***cylinder*** is a ruled surface where all the rulings are parallel in $\mathbb{R}^3$. For an example Let $\boldsymbol{c}(s) = (x(s), y(s))$ be a curve in $\mathbb{R}^2$. Then

$$\boldsymbol{x}(s, z) = (x(s), y(s), z)$$

is a cylinder, with all the rulings being parallel to the z-axis.

Problem 1. Not all examples are so obvious. For the surface, M, defined by

$$x^2 + y^2 - z^2 = 1$$

show that for any $\alpha \in \mathbb{R}$ both of the lines

$$\boldsymbol{c}_{\alpha,\pm}(t) = (\cos(\alpha) - t\sin(\alpha), \sin(\alpha) + t\cos(\alpha), \pm t)$$

are on M. Thus M is note only ruled but is ***doubly rule***. *Hint:* One way to do this is just the obvious plug and chug. But as this is at least partly a geometry class, let us try to give a more geometry argument. The surface

M is invariant under rotations about the z axis, a fact we will consider geometrically evident and you do not have the prove. The rotation by α about the z axis has matrix

$$R(\alpha) = \begin{bmatrix} \cos(\alpha) & -\sin(\alpha) & 0 \\ \sin(\alpha) & \cos(\alpha) & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

(Again you do not have to prove this.) When $\alpha = 0$ these the lines simplify to

$$\boldsymbol{c}_{\pm}(t) = \begin{bmatrix} 1 \\ t \\ \pm t \end{bmatrix}$$

and it is more or less obvious (or at least an easy calculation) to see these are on M. (We have written this as a column vector as this is what plays well with matrix multiplication.) As $R(\alpha)$ is linear, the curves $R(\alpha)\boldsymbol{c}_{\pm}$ are lines in $\mathbb{R}^3$ and Since M is invariant under $R(\alpha)$ the lines $R(\alpha)\boldsymbol{c}_{\pm}$ on M. Now just check that

$$R(\alpha)\boldsymbol{c}_{\pm}(t) = \boldsymbol{c}_{\alpha,\pm}(t)$$

(again writing the vectors as columns). □

Problem 2. For a more challenging problem show that the surface $z = xy$ is also doubly ruled. Feel free to ask for a hint in class if starts to look too complicated. □

A general ruled surface can be parameterized as

$$\boldsymbol{x}(u, v) = \boldsymbol{c}(v) + u\boldsymbol{b}(v)$$

There are several cases. The first is that $\boldsymbol{b}' \equiv 0$. Then $\boldsymbol{b}$ is constant and we have seen that in appropriate coordinates this implies

$$\boldsymbol{x}(u, v) = (x(v), y(v), u)$$

or in what maybe more natural coordinates

$$\boldsymbol{x}(s, z) = (x(s), y(s), z)$$

and we can choose the curve $(x(s), y(s))$ to be unit speed.

So form now on we assume that $\boldsymbol{b}'(v) \neq \boldsymbol{0}$ for all v. We have shown that we can choose the curve $\boldsymbol{c}$ with the property

$$\langle \boldsymbol{c}'(v), \boldsymbol{b}'(v) \rangle = 0 \qquad (1)$$

along $\boldsymbol{c}$. This curve is the ***line of striction*** and is uniquely defined (up to reparameterization) by (1).

A case where (1) holds is when $\boldsymbol{c}'(v) \equiv \boldsymbol{0}$. That is when $\boldsymbol{c}(v)$ is constant. Then by translation we can assume that $\boldsymbol{c} = \boldsymbol{0}$ is the origin of $\mathbb{R}^3$. Then

$$\boldsymbol{x}(u, v) = u\boldsymbol{b}(v).$$

We can reparameterize $\boldsymbol{b}$ to be unit speed.

Problem 3. Let $\boldsymbol{b}\colon [a,b] \to S^2$ (where $S^2 = \{\boldsymbol{v} \in \mathbb{R}^3 : \|\boldsymbol{v}\| = 1\}$) be a unit speed curve and define $\boldsymbol{x}\colon \mathbb{R} \times [a,b] \to \mathbb{R}^3$ by

$$\boldsymbol{x}(u,v) = u\boldsymbol{b}(v).$$

(a) We have

$$\boldsymbol{x}_u = \boldsymbol{b}(v), \qquad \boldsymbol{x}_v = u\boldsymbol{b}(v).$$

Note that as $\boldsymbol{b}(v)$ and $\boldsymbol{b}'(v)$ are both unit vectors and $\boldsymbol{b} \perp \boldsymbol{b}'$ (which follows from by taking the derivative of $\langle \boldsymbol{b}, \boldsymbol{b} \rangle \equiv 1$) and thus the cross product $\boldsymbol{b} \times \boldsymbol{b}'$ is a unit vector. Use this to show

$$\boldsymbol{n}(u,v) = \boldsymbol{b}(v) \times \boldsymbol{b}'(v)$$

is a unit normal to the surface.

(b) Show that if S is the shaper operator defined by $\boldsymbol{n}$, that

$$S\boldsymbol{x}_u = \boldsymbol{0}$$

and therefore one principal curvature $\kappa_1 = 0$ and $\boldsymbol{x}_u$ is a principal direction of M.

(c) Show

$$S\boldsymbol{x}_v = -\frac{\partial}{\partial v}\boldsymbol{n} = -\boldsymbol{b}(v) \times \boldsymbol{b}''(v).$$

But, as $\boldsymbol{b}$ is unit speed the we have $\boldsymbol{b}'' \perp \boldsymbol{b}, \boldsymbol{b}'$ and therefore $\boldsymbol{b}(v) \times \boldsymbol{b}'(v) = \lambda(v)\boldsymbol{b}''(v)$ for some scalar function λ. Use that $\boldsymbol{b}'(v)$ is a unit vector to show

$$\lambda(v) = \langle \boldsymbol{b}'(v), \boldsymbol{b}(v) \times \boldsymbol{b}'' \rangle$$

and thus

$$S\boldsymbol{x}_v = -\lambda(v)\boldsymbol{b}'(v).$$

(d) Use what we have just shown to show

$$S\boldsymbol{x}_v = -\frac{\lambda(v)}{u}\boldsymbol{x}_v$$

and thus that the other principal curvature is

$$\kappa_2(v) = -\frac{\lambda(v)}{u}$$

and that $\boldsymbol{x}_v$ is a principal direction of M. □

So we now add the assumption that

$$\boldsymbol{c}'(v) \neq 0$$

in our parameterization

$$\boldsymbol{x}(u,v) = \boldsymbol{c}(s) + u\boldsymbol{b}(v).$$

Then

$$\boldsymbol{x}_u = \boldsymbol{b}(v), \qquad \boldsymbol{x}_v = \boldsymbol{c}'(v) + u\boldsymbol{b}'(v).$$

Thus

$$\boldsymbol{x}_u \times \boldsymbol{x}_v = \boldsymbol{b}(v) \times \boldsymbol{c}'(v) + u\boldsymbol{b}(v) \times \boldsymbol{b}'(v).$$

By our assumption that $\boldsymbol{c}$ is the line of striction we have $\boldsymbol{c}' \perp \boldsymbol{b}'$ and as $\|\boldsymbol{b}\| = 1$ we also have $\boldsymbol{b} \perp \boldsymbol{b}'$. Therefore $\boldsymbol{b}'$ is orthogonal to both of $\boldsymbol{b}$ and $\boldsymbol{c}'$ and therefore it points in the same direction as $\boldsymbol{c}' \perp \boldsymbol{b}'$ (as $\boldsymbol{c}' \perp \boldsymbol{b}'$ is also orthogonal to both $\boldsymbol{b}$ and $\boldsymbol{c}'$). Therefore

$$\boldsymbol{b}(v) \times \boldsymbol{c}'(v) = \lambda(v)\boldsymbol{b}'(v)$$

for some scaler valued function λ (which is not the same as the λ above). Thus we now have

$$\boldsymbol{x}_u \times \boldsymbol{x}_v = \lambda(v)\boldsymbol{b}'(v) + u\boldsymbol{b}(v) \times \boldsymbol{b}'(v).$$

Note that $\boldsymbol{b}'$ and $\boldsymbol{b} \times \boldsymbol{b}'$ are orthogonal and $\boldsymbol{b}$ is a unit vector. Therefore

$$\|\boldsymbol{x}_u \times \boldsymbol{x}_v\|^2 = \lambda(v)^2\|\boldsymbol{b}'\|^2 + u^2\|\boldsymbol{b}(v) \times \boldsymbol{b}'(v)\|^2 = (\lambda(v)^2 + u^2)\|\boldsymbol{b}'(v)\|^2.$$

Problem 4. Recall that a parameterization, $\boldsymbol{x}$, of a surface is regular if and only if $\boldsymbol{x}_u \times \boldsymbol{x}_v \neq \mathbf{0}$. We are assuming $\boldsymbol{b}' \neq \mathbf{0}$ and so we have $\boldsymbol{x}_u \times \boldsymbol{x}_v \neq 0$ if and only if

$$\lambda(v)^2 + u^2 \neq 0.$$

Show this only happens when $u = 0$ (which means we are on the line of striction) and $\lambda(v) = 0$. Also show $\lambda(v) = 0$ if and only if $\boldsymbol{b}(v) \times \boldsymbol{c}'(v) = \mathbf{0}$. □

Problem 5. So we now have anther special case to consider, that is $\boldsymbol{c}' \neq \mathbf{0}$, but $\boldsymbol{c}' \times \boldsymbol{b} \equiv \mathbf{0}$. If this is the case it means that $\boldsymbol{c}'$ and $\boldsymbol{b}$ point in the same direction. Thus we can parameterize M by

$$\boldsymbol{y}(s,t) = \boldsymbol{c}(s) + t\boldsymbol{c}'(s).$$

Such a surface is called a ***tangent developable***. Show this has Gauss curvature, $K = 0$ at all point not on $\boldsymbol{c}$. *Hint:* Assume that $\boldsymbol{c}$ is unit speed and use the Frenet formulas. □