Notes on hyperbolic functions.

The hyperbolic functions cosh and sinh are generally introduced by

$$\cosh(x) = \frac{e^x + e^{-x}}{2}$$
$$\sinh(x) = \frac{e^x - e^{-x}}{2}$$

and most of the properties of these function determined directly from the properties of e^x. Here I will show it is as easy, maybe easier to deduce the properties of the hyperbolic functions from the fact they are a fundamental set of solutions for a differential equation. It follows from the definitions above that

$$\cosh''(x) = \cosh(x) \qquad \cosh(0) = 1 \qquad \cosh'(0) = 0$$
$$\sinh''(x) = \sinh(x) \qquad \sinh(0) = 0 \qquad \sinh'(0) = 0$$

The following is a special case of a much more general result uniqueness about linear second order differential equations.

Theorem 1. *Let $f\colon \mathbb{R} \to \mathbb{R}$ be a twice differentiable function that satisfies*

$$f''(x) = f(x)$$

then is a linear combination of cosh *and* sinh, *more exactly*

$$f(x) = f(0)\cosh(x) + f'(0)\sinh(x).$$

□

Proposition 2. *The derivatives of* cosh *and* sinh *are*

$$\cosh'(x) = \sinh(x)$$
$$\sinh'(x) = \cosh(x)$$

Also cosh *is an even function and* sinh *is an odd function:*

$$\cosh(-x) = \cosh(x), \qquad \sinh(-x) = -\sinh(x).$$

Proof. While this is follows easily from the definitions I prove it form Theorem 1 as warm up for more complicated formulas. Let $f = \cosh$. Then

$$f'' = f, \qquad f(0) = 0, \quad f'(0) = 1.$$

Let $g = f'$. Then

$$g'' = (f')'' = f''' = (f'')' = f' = g$$

and

$$g(0) = f'(0), \qquad g'(0) = f''(0) = f(0) = 1.$$

Therefore Theorem 1 implies

$$g(x) = g(0)\cosh(x) + g'(0)\sinh(x) = 0\cosh(x) + 1\sinh(x) = \sinh(x).$$

As $g = f' = \cosh'$ we have shown $\cosh' = \sinh$.

Likewise if we set $g = \sinh'$ similar calculations show $g'' = g$, $g(0) = 1$, and $g'(0) = 0$ and therefore Theorem gives

$$\sinh'(x) = g(0)\cosh(x) + g'(0)\sinh(x) = 1\cosh(x) + 0\sinh(x) = \cosh(x).$$

Let $f(x) = \cosh(-x)$. Then $f'(x) = -\sinh(-x)$ and $f''(x) = \cosh(-x) = f(x)$. Thus $f''(x) = f(x)$, $f(0) = \cosh(0) = 1$ and $f'(0) = -\sinh(0) = 0$. Thus, Theorem 1 again,

$$\cosh(-x) = f(x) = f(0)\cosh(x) + f'(0)\sinh(x) = \cosh(x).$$

If $f(x) = \sinh(-x)$, then again $f'' = f$, but this time $f(0) = 0$ and $f'(0) = -\cosh(-0) = -1$ and

$$\sinh(-x) = f(x) = f(0)\cosh(x) + f'(0)\sinh(x) = -\sinh(x).$$

□

Proposition 3. *The identity*

$$\cosh^2(x) - \sinh^2(x) = 1$$

holds.

Proof. Let

$$E(x) = \cosh^2(x) - \sinh^2(x).$$

Then by the derivative formulas

$$\begin{aligned} kE'(x) &= 2\cosh(x)\cosh'(x) - 2\sinh(x)\sinh'(x) \\ &= 2\cosh(x)\sinh(x) - 2\sinh(x)\cosh(x) \\ &= 0 \end{aligned}$$

Thus E is constant. As $E(0) = \cosh^2(0) - \sinh^2(0) = 1$, the constant value is 1, as required. □

This implies the hyperbola

$$x^2 - y^2 = 1$$

is parameterized by

$$\mathbf{c}(t) = (\cosh(t), \sinh(t)).$$

More generally

$$\frac{(x - x_0)^2}{a^2} - \frac{(y - y_0)^2}{b^2} = 1$$

is parameterized by

$$\begin{aligned} x(t) &= x_0 + a\cosh(t) \\ y(t) &= y_0 + b\sinh(t) \end{aligned}$$

Proposition 4. *The addition formulas*

$$\begin{aligned} \cosh(a + b) &= \cosh(a)\cosh(b) + \sinh(a)\sinh(b) \\ \sinh(a + b) &= \sinh(a)\cosh(b) + \cosh(a)\sinh(b) \end{aligned}$$

hold.

Proof. Let

$$f(x) = \cosh(a + x).$$

Then

$$f''(x) = \cosh''(a + x) = \cosh(a + x) = f(x)$$

and

$$f(0) = \cosh(a), \qquad f'(0) = \cosh'(a) = \sinh(a).$$

Therefore by Theorem 1

$$\begin{aligned} \cosh(a + x) = f(x) &= f(0)\cosh(x) + f'(0)\sinh(x) \\ &= \cosh(a)\cosh(x) + \sinh(a)\cosh(x) \end{aligned}$$

Letting $= a$ gives the addition for cosh. A similar argument using $f(x) = \sinh(a + x)$ gives the addition formula for sinh. □

Corollary 5. *The double angle formulas hold:*

$$\cosh(2x) = \cosh^2(x) + \sinh^2(x)$$
$$\sinh(2x) = 2\sinh(x)\cosh(x).$$

Proof. Let $a = b = x$ in the addition formulas. □

Using $\cosh^2 - \sinh^2 = 1$ the double angle formula for cosh can also be written as

$$\cosh(2x) = 2\cosh^2(x) - 1 = 1 + 2\sinh^2(x).$$

This implies the hyperbolic versions of the half angle formulas. Rearranging these gives

$$\cosh^2(x) = \frac{\cosh(2x) + 1}{2}$$
$$\sinh^2(x) = \frac{\cosh(2x) - 1}{2}$$

Sometimes the version of these with x replaced by $x/2$,

$$\cosh^2(x/2) = \frac{\cosh(x) + 1}{2}$$
$$\sinh^2(x/2) = \frac{\cosh(x) - 1}{2}$$

can be of use when doing integrals.

To finish up these notes we do some integrals using hyperbolic substations. Some of these can also be done with trigonometric substations. To start note that the derivative formulas for cosh and sinh imply

$$\int \cosh(x)\,dx = \sinh(x) + C$$
$$\int \sinh(x)\,dx = \cosh(x) + C$$

Example 6. Compute $\int \sqrt{x^2+1}\,dx$. Let $x = \sinh(t)$. Then $dx = \cosh(t)\,dt$. Therefore (using various of the identities above)

$$\begin{aligned}
\int \sqrt{x^2+1}\,dx &= \int \sqrt{1+\sinh^2(t)}\ \cosh(t)\,dt \\
&= \int \sqrt{\cosh^2(t)}\ \cosh(t)\,dt \\
&= \int \cosh^2(t)\,dt \\
&= \int \left(\frac{\cosh(2t)+1}{2}\right)\,dt \\
&= \frac{1}{2}\left(\frac{\sinh(2t)}{2}+t\right)+C \\
&= \frac{1}{2}\left(\frac{2\sinh(t)\cosh(t)}{2}+t\right)+C \\
&= \frac{1}{2}\left(\sinh(t)\cosh(t)+t\right)+C \\
&= \frac{1}{2}\left(x\sqrt{x^2+1}+\operatorname{arcsinh}(x)\right)+C
\end{aligned}$$

where we have used that $x = \sinh(t)$ implies $\cosh(t) = \sqrt{x^2+1}$.

Example 7. Compute the integral $\int \sqrt{\frac{x+1}{x-1}}\,dx$. To start use the formulas for $\cosh^2(x)$ and $\sinh^2(x)$ above to get

$$\frac{\cosh(t)+1}{\sinh(t)-1} = \frac{\cosh^2(t/2)}{\sinh^2(t/2)}.$$

This suggests doing the substation $x = \cosh(t)$, $dx = \sinh(t)\, dt$ and again using several of the formulas above

$$\begin{aligned}
\int \sqrt{\frac{x+1}{x-1}}\, dx &= \int \sqrt{\frac{\cosh^2(t/2)}{\sinh^2(t/2)}}\, \sinh(t)\, dt \\
&= \int \left(\frac{\cosh(t/2)}{\sinh(t/2)}\right) \sinh(t)\, dt \\
&= \int \left(\frac{\cosh(t/2)}{\sinh(t/2)}\right) 2\sinh(t/2)\cosh(t/2)\, dt \\
&= 2\int \cosh^2(t/2)\, dt \\
&= 2\int \left(\frac{\cosh(t)+1}{2}\right)\, dt \\
&= \sinh(t) + t + C \\
&= \sqrt{x^2-1} + \operatorname{arccosh}(x) + C.
\end{aligned}$$