SHUR TYPE COMPARISON THEOREMS FOR AFFINE CURVES WITH APPLICATION TO LATTICE POINT ESTIMATES

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ABSTRACT. If $c, \overline{c}\colon [a,b] \to \mathbb{R}^2$ are two convex planar curve parameterized by affine arc length and $A\colon [a,b] \to [0,\infty)$ is the area bounded by the restriction $c\big|_{[a,s]}$ and the segment between $c(a)$ and $c(s)$ with $\overline{A}$ the corresponding function for $\overline{c}$, and the affine curvature are related by $\varkappa(s) \le \overline{\varkappa}(s)$, then $A(s) \ge \overline{A}(s)$. Also for any point of a convex curve we define *adapted affine coordinates* centered at the point and give sharp estimates on the coordinates of the curve in terms of bounds on the curvature. Proving these bounds involves generalizing classical comparison theorems of Strum-Liouville type to higher order and nonhomogenous equations. These estimates allow us to give sharp bounds on the areas of inscribed triangles in terms of affine curvature and the affine distance between the vertices. These inequalities imply upper bounds of the number of lattice points on a convex curve in terms of its affine arc length.

1. Introduction.

The results in this paper were motivated first by wanting to give extensions, or more precisely analogues, of some of the results of the Euclidean differential geometry of curves to the affine setting;. The second motivation was to use these results to give upper bounds on the number of lattice points on a convex curve in terms of the affine arc length and bounds on affine curvature of the curve.

Let $\mathcal{C}$ be an embedded connected curve in the plane. Then we call $\mathcal{C}$ a ***convex*** if and only if $\mathcal{C}$ is on the boundary of its convex hull. The curve is a ***closed convex*** curve if and only if it is the boundary of a bounded convex set.

A basic result in the Euclidean differential geometry of curves is the comparison theorem of A. Schur: if $\mathcal{C}$ and $\overline{\mathcal{C}}$ are two convex non-closed curves of the same length and the curvature of $\overline{\mathcal{C}}$ is pointwise greater

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than the curvature of $\mathcal{C}$, then the distance between the endpoints of $\overline{\mathcal{C}}$ is smaller than the distance between the endpoints of $\mathcal{C}$. That is the greater the curvature the smaller the distance between the endpoints. (See [3, Page 36] or [6, Thm. 2-19, Page 31] for the precise statement and a proof.) The original proof is in [13]. Some Some extensions and generalization are given in [15] (higher dimensions and minimal regularity), [5] (version in hyperbolic space) and [11] (version in the Lorentzian plane).

In affine geometry the distance between points in the plane is not defined, but the area bounded by the segment joining two points on the curve and the curve (see Figure 1) is defined. Recall a ***special affine motion*** of $\mathbb{R}^2$ is a map of the form $v \mapsto Mv+b$ where M is a linear map with $\det(M) = 1$ and $b \in \mathbb{R}^2$. (See Section 2 for our conventions and definitions of affine arc length and affine curvature.) The affine version of Shur's Theorem is the greater curvature the smaller the bounded area:

Theorem A. *Let $c, \overline{c} \colon [a,b] \to \mathbb{R}^2$ be curves parameterized by affine arc length with affine curvatures $\varkappa$ and $\overline{\varkappa}$ respectively and let A and $\overline{A}$ be the respective areas bounded by the curves and the segment between their endpoints as in Figure 1 (see Section 2 for the precise definition of A and $\overline{A}$). Assume that $\overline{\varkappa}$ satisfies any one of the following three conditions (a) $\overline{\varkappa}$ is constant, (b) $\overline{\varkappa} \le 0$, or (c) $\overline{\varkappa} \le k_1$ where k_1 is a positive constant with $k_1 \le (\pi/(b-a))^2$. Then*

$$\begin{aligned} \varkappa \le \overline{\varkappa} \text{ on } [a,b] \quad &\text{implies} \quad A \ge \overline{A} \\ \varkappa \ge \overline{\varkappa} \text{ on } [a,b] \quad &\text{implies} \quad A \le \overline{A} \end{aligned}$$

In either case $A = \overline{A}$ implies c is the image of $\overline{c}$ by a special affine motion.

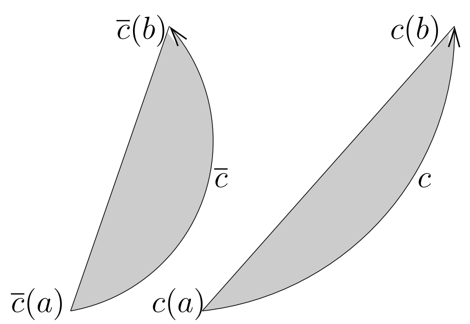

Figure 1. The curves c and $\overline{c}$ showing the areas bounded by the curves and the secant segments between their endpoints.

This is a special case of Theorem 4.1 below. Besides this result we give comparison results for the areas of inscribed triangles (Proposition 4.3 Theorem 5.8) and coordinate systems centered at point on the curve that are "adapted" to the geometry of the curve (Theorem 5.5).

These results are based on generalizations of Sturm type comparison theorems for homogeneous second order differential differential equations to equations that are non-homogeneous and of higher order. To give an example of the relevance of such results, in Theorem 1 let $A = A(b)$ be viewed as a function of the end point $c(b)$, then a special case of Theorem 2.2, is that A satisfies the third order initial value problem

$$A''' + \varkappa A' = \frac{1}{2}, \qquad A(a) = A'(a) = A''(a) = 0.$$

Thus Theorem A follows from a comparison theorem for differential equations of this form. Section 3 has a general theory for comparisons of solutions to linear ordinary differential equations.

Finally we consider the problem of bounding the number of lattice points on a convex curve by bounds its affine arc length and bounds on its affine curvature. The original result about the number of lattice points on a convex curve is result of Jarník [9] that a closed strictly convex curve of length L contains at most $3(2\pi)^{-1/3}L^{2/3} + O(L^{1/3})$ and the exponent and constant of the leading term are best possible. There have been improvements of Jarník's result which involve bounds of the higher derivatives of the curve by Bombieri and Pila[2] and Swinnerton-Dyer [16] and others. In the case of bounds on the fourth derivative it is natural to interpret the bounds on the derivatives in terms of affine curvature and to give the bounds in terms of the affine arc length. In the case of arcs on ellipses this was done in [8]. Anther reason affine arc length more natural than Euclidean arc length for these questions is that the lattice $\mathbb{Z}^2$ in variant under the group $\mathrm{SL}\, 2, \mathbb{Z}$ of 2×2 integer matrices of determent one. Affine arc length is invariant under this group, while Euclidean arc length is not.

Theorem B. *Let k_0 and k_1 be constants with $k_0 \leq k_1$. Then there is a number $L_{k_0} > 0$ and so that if $k_1 \leq (\pi/(2L_{k_0}))^2$ (which is automatic if $k_1 \leq 0$) so that if $\mathcal{C}$ is a convex curve whose affine curvature satisfies $k_0 \leq \varkappa \leq k_1$ and $m = \lfloor \Lambda(\mathcal{C})/(2L_{k_0}) \rfloor$ then*

$$\#(\mathbb{Z}^2 \cap \mathcal{C}) \leq 2m + 2.$$

When $k_0 = 0$ the constant is $L_0 = 1$. In this case the example of the parabolic arc $\mathcal{C} = \{(s, s(s-1)/2) : 0 \leq s \leq 2m+1\}$ shows this estimate is sharp. For $k_0 < 0$ the number $L = L_{k_0}$ is the solution to

$$\left(\frac{\sinh(\sqrt{|k_0|}\,L)}{\sqrt{|k_0|}}\right)\left(\frac{\cosh(\sqrt{|k_0|}\,L) - 1}{|k_0|}\right) = \frac{1}{2}.$$

In this case there are also examples where the bound is sharp, see Section 7. There is a similar equation for k_0 when $k_0 > 0$. However the discussion and examples in Paragraph 7.4 show these results are of greatest interest when $k_0 \leq 0$.

2. Affine arc length, affine curvature, and the differential equation for the area function.

We give our conventions on the geometry of affine curves. Other sources for this material are [1], [6], and [14]. If $v, w \in \mathbb{R}^2$ then $v \wedge w = v_1 w_2 - v_2 w_1$ where $v = (v_1, v_2)$ and $w = (w_1, w_2)$. This is the determinant of the matrix with rows v and w. If I is an interval and $\gamma\colon I \to \mathbb{R}^2$ is a C^2 curve, where the velocity vector $\gamma'(t)$ and the acceleration vector $\gamma''(t)$ are linearly independent for all $t \in I$, then a direct calculation shows the one form

$$(\gamma'(t) \wedge \gamma''(t))^{1/3}\, dt$$

is invariant under both C^2 reparameterizations and special affine motions of $\mathbb{R}^2$. The ***affine arc length*** of c, denoted $\Lambda(c)$, is defined by integrating this one form. Thus $\Lambda(c) = \int_a^b (c'(t) \wedge c''(t))^{1/3}\, dt$ where $[a, b]$ is the domain of c. Equivalently if $c\colon I \to \mathbb{R}^2$, then s is affine arc length along c if and only if

$$c'(s) \wedge c''(s) \equiv 1.$$

This gives the curve has a natural orientation: the orientation for which the basis $c'(s)$, $c''(s)$ is always positive (that is a right handed) along c. This is the same as the orientation that makes the one form $c'(t) \wedge c''(t)^{\frac{1}{3}}\, dt$ positive. The vector $c'(s)$ is the ***affine tangent vector*** and $c''(s)$ is the ***affine normal vector***. With this orientation the Euclidean curvature, κ, is positive which implies c is locally convex in the sense that at any point its tangent line is a locally a support line in that near the point the curve lines in the closed half plane bounded by the tangent line and with the affine normal pointing into the half plane.

Taking the derivative of $c'(s) \wedge c''(s) = 1$ and using $c'(s) \wedge c'(s) = 0$ gives

$$c'(s) \wedge c'''(s) = 0.$$

This implies $c'''(s)$ is linearly dependent on $c'(s)$ and therefore for some scalar function $\varkappa(s)$,

$$c'''(s) = -\varkappa(s)c'(s).$$

The function $\varkappa$ is the ***affine curvature*** of c. (This sign is chosen so that ellipses have constant positive affine curvature and hyperbolas have constant negative curvature.) The affine curvature determines a curve up to an affine motion.

Theorem 2.1. *Let I be in interval in $\mathbb{R}$ and $c_1, c_2 \colon I \to \mathbb{R}^2$ be C^3 affine unit speed curves that have the same affine curvature at each point. Then c_1 and c_2 differ by an affine motion. That is for some linear map $M \colon \mathbb{R}^2 \to \mathbb{R}^2$ with $\det(M) = 1$ and some $b \in \mathbb{R}^2$ where holds $c_2(s) = Mc_1(s) + b$ for all $s \in I$.* □

Proofs can be found in [1], [6], and [14]. It is worth remarking that while this theorem only requires $c_1(s)$ and $c_2(s)$ to be C^3 functions of the affine arc length, the images of these curves will be C^4 immersed submanifold of $\mathbb{R}^2$. See Proposition 5.4

If $c \colon I \to \mathbb{R}^2$ for some interval I, $a \in I$ and $p_0 \in \mathbb{R}^2$, define

$$A_{c,a,p_0}(s) := \begin{cases} \text{Signed area bounded by the restriction } c\big|_{[a,s]} \\ \text{and the segments } \overline{p_0 c(a)} \text{ and } \overline{p_0 c(s)}. \end{cases}$$

See Figure 2. To give precise and more computationally useful definition let f be the function

$$f(t, s) = (1 - t)p_0 + tc(s), \qquad 0 \le t \le 1, \quad s \in I.$$

Then $A_{c,a,p_0}(s)$ is the signed area of the set of points $\{f(t, \sigma) : 0 \le t \le 1 \text{ and } \sigma \text{ between } a \text{ and } s\}$, which is a quantity we can compute with an integral. The partial derivatives and Jacobian of f are

$$\begin{aligned} \frac{\partial f}{\partial t} &= c(s) - p_0 \\ \frac{\partial f}{\partial s} &= tc'(s) \\ \frac{\partial f}{\partial t} \wedge \frac{\partial f}{\partial s} &= t(c(s) - p_0) \wedge c'(s). \end{aligned}$$

So a precise definition of A_{c,a,p_0} is

$$A_{c,a,p_0}(s) = \int_a^s \int_0^1 \frac{\partial f}{\partial t}(t,\sigma) \wedge \frac{\partial f}{\partial s}(t,\sigma)\, dt\, d\sigma \tag{2.1}$$
$$= \int_a^s \int_0^1 t(c(\sigma) - p_0) \wedge c'(\sigma)\, dt\, d\sigma$$
$$= \frac{1}{2}\int_a^s (c(\sigma) - p_0) \wedge c'(\sigma)\, d\sigma.$$

In computing the area $A_{c,a,p_0}(s)$ the points where $f_t \wedge f_u = t(c(\sigma) - p_0) \wedge c'(\sigma)$ is positive the area is counted as positive, and when this is negative the area is negative the area is negative.

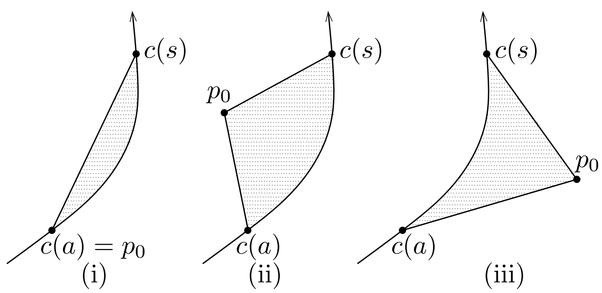

Figure 2. The area $A_{c,a,p_0}(s)$ for some choices of the point p_0. In (i) and (ii) we have $A_{c,a,p_0}(s) > 0$ and in (iii) $A_{c,a,p_0}(s) < 0$.

By taking the first three derivatives of $A(s) := A_{c,a,p_0}(s)$ and using $c'(s) \wedge c''(s) = 1$ and $c'''(s) = -\varkappa(s)c(s)$ we find it satisfies a third order differential equation.

$$A'(s) = \frac{1}{2}(c(s) - p_0) \wedge c'(s) \tag{2.2}$$

$$A''(s) = \frac{1}{2}c'(s) \wedge c'(s) + \frac{1}{2}(c(s) - p_0) \wedge c''(s) \tag{2.3}$$
$$= \frac{1}{2}(c(s) - p_0) \wedge c''(s)$$

$$A'''(s) = \frac{1}{2}c'(s) \wedge c''(s) + \frac{1}{2}(c(s) - p_0) \wedge c'''(s) \tag{2.4}$$
$$= \frac{1}{2} + \frac{1}{2}(c(s) - p_0) \wedge (-\varkappa(s)c'(s))$$
$$= \frac{1}{2} - \varkappa(s)A'(s).$$

This proves:

Theorem 2.2. *For any point p_0 the area function $A(s) = A_{c,a,p_0}(s)$ satisfies the third order differential equation*

$$A''' + \varkappa A' = \frac{1}{2}.$$

with initial conditions

$$A(a) = 0, \quad A'(a) = \frac{1}{2}(c(a) - p_0) \wedge c'(a), \quad A''(a) = \frac{1}{2}(c(a) - p_0) \wedge c''(a).$$

□

2.1. **Some notation and conventions.** Some of our results are awkward to state for curves given with a particular parameterization on an interval, that is as $c\colon I \to \mathbb{R}^2$. So we will use $\mathcal{C}$ and subscripted variants to denote an embedded convex curve. If $p \in \mathcal{C}$ then $\mathbf{t}_{\mathcal{C}}(p)$ and $\mathbf{n}_{\mathcal{C}}(p)$ will be the affine tangent and affine normal. Explicitly if $c\colon I \to \mathcal{C}$ is a local affine unit speed local parameterization of $\mathcal{C}$ with $c(s_0) = p$, then $\mathbf{t}_{\mathcal{C}}(p) = c'(s_0)$ and $\mathbf{n}_{\mathcal{C}}(s_0) = c''(s_0)$. The affine curvature of $\mathcal{C}$ at p is denoted $\varkappa(p)$.

3. Comparison results for initial value problems.

Definition 3.1. Let I be an interval in $\mathbb{R}$ and let $\mathcal{D}$ be a linear differential operator defined on $C^n(I)$ by

$$\mathcal{D}y = \frac{d^n y}{ds^n} + \sum_{j=0}^{n-1} a_j(s) \frac{d^j y}{ds^j}$$

where the functions $a_0, \ldots, a_{n-1}$ are continuous on I. Then the ***Lagrange kernel*** for $\mathcal{D}$ on this interval is the function $K\colon I \times I \to \mathbb{R}$ defined by the initial value problem

$$\frac{\partial^n K}{\partial s^n}(s; r) + \sum_{j=0}^{n-1} a_j(s) \frac{\partial^j K}{\partial s^j}(s; r) = 0 \tag{3.1}$$

$$\frac{\partial^j K}{\partial s^j}(r; r) = 0 \quad \text{for} \quad 0 \le j \le n-2, \qquad \frac{\partial^{n-1} K}{\partial s^{n-1}}(r; r) = 1. \tag{3.2}$$

Holding r fixed the equation (3.2) is a linear ordinary differential equation for the function $s \mapsto K(s; r)$ so the existence and uniqueness of K follows from the existence and uniqueness theorem for linear ordinary differential equations. The continuity of $K(s; r)$ follows from results on the continuous dependence of solutions of differential equations on initial conditions and parameters (cf. [4, Chapter 1]).

Remark 3.2. There does not seem to be a standard name for this kernel. Some authors refer to it as a Green's function (e.g. [10]), but as this term is usually reserved for boundary value problems rather than initial value problems this seems a little misleading. As it is used to solve the Cauchy problem for the non-homogeneous $\mathcal{D}y = f$ some authors (e.g. [7]) refer to it as the Cauchy kernel. But as it is just the kernel one gets by applying Lagrange's method of variation of parameters (cf. [10, Pages 145–147]) it seems likely that Lagrange was the first to write down some form of it.

Theorem 3.3. *Let $\mathcal{D}$ be as in Definition 3.1, the function $f\colon I \to \mathbb{R}$ continuous, and $r \in I$. Then, denoting the j-th derivative of y by $y^{(j)}$, the solution to the initial value problem*

$$\mathcal{D}y = f, \qquad y^{(j)}(r) = 0 \quad \textit{for} \quad j = 0, 1, \ldots, n-1$$

is

$$y(s) = \int_r^s K(s;t) f(t)\, dt.$$

Proof. This is known (and seems to be somewhat of a folk theorem among some applied mathematicians) and a proof of a somewhat stronger version can be found in [10, Theorem 4-2, Page 149]. We include a short proof for completeness. From the fundamental theorem of calculus and Leibniz's formula for differentiating the integral of a function depending on a parameter

$$\frac{\partial}{\partial s}\int_r^s \frac{\partial^j K}{\partial s^j}(s;t) f(t)\, dt = \frac{\partial^j K}{\partial s^j}(s;s) f(s) + \int_r^s \frac{\partial^{j+1} K}{\partial s^{j+1}}(s;t) f(t)\, dt \tag{3.3}$$

holds for $j = 0, \ldots, (n-1)$. When $j \le n-2$ we have $(\partial^j K/\partial s^j)(s;s) = 0$ and this gives

$$\frac{\partial}{\partial s}\int_r^s \frac{\partial^j K}{\partial s^j}(s;t) f(t)\, dt = \int_r^s \frac{\partial^{j+1} K}{\partial s^{j+1}}(s;t) f(t)\, dt$$

and therefore for $j = 0, 1, \ldots, n-1$

$$y^{(j)}(s) = \int_r^s \frac{\partial^j K}{\partial s^j}(s;t) f(t)\, dt.$$

When $j = n-1$, using $(\partial^{n-1}K/\partial s^{n-1})(s;s) = 1)$ and the differential equation for K,

$$\begin{aligned} y^{(n)}(s) &= f(s) + \int_r^s \frac{\partial^n K}{\partial s^n}(s;t)f(t)\,dt \\ &= f(s) + \int_r^s \left(-\sum_{j=0}^{n-1} a_j(s) \frac{\partial^j K}{\partial s^j}(s;t) \right) f(t)\,dt \\ &= f(s) - \sum_{j=0}^{n-1} a_j(s) \int_r^s \frac{\partial^j K}{\partial s^j}(s;t)f(t)\,dt \\ &= f(s) - \sum_{j=0}^{n-1} a_j(s) y^{(j)}(s) \end{aligned}$$

and therefore $\mathcal{D}y = f$. The initial conditions $y^{(j)}(r) = 0$ for $j = 0, 1, , \dots, n-1$ follow from $\int_r^r = 0$. □

Definition 3.4. Let I be an interval and $\mathcal{D}$ a linear differential operator as in Definition 3.1 and let K be the Lagrange kernel of $\mathcal{D}$. Then K is ***forward positive*** on I if and only if for all $r, s \in I$

$$s > r \quad \text{implies} \quad K(s;r) \geq 0.$$

Remark 3.5. If K is forward positive in this sense, then for a fixed r the function given by $y(s) := K(s;r)$ is a not identically zero solution to a homogeneous linear differential equation. The zeros of a such a solution are isolated. Thus for fixed r, the zero set of $s \mapsto K(s;r)$ is a discrete set. Therefore when K is forward positive $s \mapsto K(s;r)$ is positive almost everywhere on $I \cap [r, \infty)$. This fact will be used several times.

Remark 3.6. For any linear differential operator $\mathcal{D}$ is in Definition 3.1 defined on an interval I and $r \in I$ the initial conditions (3.2) there is a $\delta > 0$ so that $K(s;r) > 0$ for $r < s < r + \delta$. Using this, continuity, and a compactness argument, it follows that for any $r \in I$ there is an interval I_0 with $r \in I_0 \subseteq I$ so that the restriction $K\big|_{I_0 \times I_0}$ is forward positive on I_0.

The definition of a Lagrange kernel being forward positive is motivated by the following comparison result. (This result can be be realized to a larger class of differential operators, but the statement of the results are awkward to state and more generality is not needed here.)

Theorem 3.7. *Let $\varkappa, \overline{\varkappa}\colon [a,b] \to \mathbb{R}$ be continuous and n and ℓ integers with $0 \le \ell < n$. Let $y, \overline{y}\colon [a,b] \to \mathbb{R}$ be C^n functions that satisfy the n-order differential equations*

$$y^{(n)} + \varkappa y^{(\ell)} = f \quad \text{and} \quad \overline{y}^{(n)} + \overline{\varkappa}\,\overline{y}^{(\ell)} = f \tag{3.4}$$

for some continuous function f and have the same initial conditions at a: $y^{(j)}(a) = \overline{y}^{(j)}(a)$ for $j = 0, 1, \dots, n-1$. Assume

(a) The Lagrange kernel, $\overline{K}$, of the differential operator $\overline{y} \mapsto \overline{y}^{(n)} + \overline{\varkappa}\,\overline{y}^{(\ell)}$ is forward positive on $[a,b]$, and
(b) $y^{(\ell)} > 0$ almost everywhere on $[a,b]$.

Then

$$\varkappa \le \overline{\varkappa} \text{ on } [a,b] \quad \text{implies} \quad y \ge \overline{y} \text{ on } [a,b], \tag{3.5}$$

$$\varkappa \ge \overline{\varkappa} \text{ on } [a,b] \quad \text{implies} \quad y \le \overline{y} \text{ on } [a,b]. \tag{3.6}$$

Moreover in either of these cases if $y(b) = \overline{y}(b)$, then $\varkappa(s) = \overline{\varkappa}(s)$ and $y(s) = \overline{y}(s)$ for all $s \in [a,b]$.

Proof. Subtract the first equation in (3.4) form the second to and rearrange a bit to get

$$(\overline{y} - y)^{(n)} + \overline{\varkappa}(y - y)^{(\ell)} = -(\overline{\varkappa} - \varkappa)y^{(\ell)}$$

As y and $\overline{y}$ have the same initial conditions at a the function $\overline{y} - y$ and its first $(n-1)$ derivatives vanish at a. Therefore Theorem 3.3 yields

$$\overline{y}(s) - y(s) = -\int_a^s \overline{K}(s;t)(\overline{\varkappa}(t) - \varkappa(t))y^{(\ell)}(t)\,dt.$$

By our assumptions for each s the inequality $K(s;t)y^{(\ell)}(t) > 0$ holds for almost all $t \in [s,b]$. Thus if $\varkappa \le \overline{\varkappa}$ on $[a,b]$ we have $y \ge \overline{y}$ on $[a,b]$, and if $y(b) = \overline{y}(b)$, then $\overline{\varkappa}(t) = \varkappa(t)$ for $t \in [a,b]$ which implies y and $\overline{y}$ stratify the same initial value problem on $[a,b]$ and thus are equal on this interval. A similar argument holds if $\varkappa \ge \overline{\varkappa}$ on $[a,b]$. □

To give the Lagrange kernel for the constant coefficient linear operators related to our problem it is convenient to introduce some notation. Let k be a real number and define functions $\mathbf{c}_k$ and $\mathbf{s}_k$ by the initial value problems

$$\begin{aligned} \mathbf{c}_k'' + k\mathbf{c}_k &= 0, & \mathbf{c}_k'(0) &= 1, & \mathbf{c}_k'(0) &= 0 \\ \mathbf{s}_k'' + k\mathbf{s}_k &= 0, & \mathbf{s}_k(0) &= 0, & \mathbf{s}_k'(0) &= 1 \end{aligned}$$

or more explicitly

$$\mathbf{c}_k(s) = \begin{cases} \cos(\sqrt{k}\, s), & k > 0; \\ 1, & k = 0; \\ \cosh(\sqrt{|k|}\, s), & k < 0. \end{cases} \quad \mathbf{s}_k(s) = \begin{cases} \dfrac{\sin(\sqrt{k}\, s)}{\sqrt{k}}, & k > 0; \\ s, & k = 0; \\ \dfrac{\sinh(\sqrt{|k|}\, s)}{\sqrt{|k|}}, & k < 0. \end{cases}$$

These satisfy

$$\mathbf{c}_k' = -k\mathbf{s}_k, \qquad \mathbf{s}' = \mathbf{c}_k \tag{3.7}$$

$$\mathbf{c}_k^2 + k\mathbf{s}_k^2 = 1 \tag{3.8}$$

$$\mathbf{c}_k(a+s) = \mathbf{c}_k(a)\mathbf{c}_k(s) - k\mathbf{s}_k(a)\mathbf{s}_k(s) \tag{3.9}$$

$$\mathbf{s}_k(a+s) = \mathbf{s}_k(a)\mathbf{c}_k(s) + \mathbf{c}_k(a)\mathbf{s}_k(s). \tag{3.10}$$

Possibly the easiest way to see the derivative formulas hold is to note $\mathbf{c}_k'$ and $-k\mathbf{s}_k$ are both solutions to the initial value problem $y'' + ky = 0$, $y(0) = 0$, $y'(0) = -k$, and $\mathbf{s}_k'$ and $\mathbf{c}_k$ are solutions to $y'' + ky = 0$, $y(0) = 1$ and $y'(0) = 0$. These imply $\mathbf{c}_k^2 + k\mathbf{s}_k^2$ has zero as its derivative and therefore is constant, this implies (3.8) holds. For the addition formula for $\mathbf{c}_k$, note the left and right hand sides of (3.9) both satisfy the initial value problem $u'' + ku = 0$, $u(0) = \mathbf{c}_k(a)$ and $u'(0) = \mathbf{c}_k'(a) = -k\mathbf{s}_k(a)$. A similar argument shows the addition formula for $\mathbf{s}_k$ holds.

Proposition 3.8. *Let $k \in \mathbb{R}$ and I an interval.*

(a) The Lagrange kernel for $y \mapsto y'' + ky$ on I is

$$P_k(s; r) = \mathbf{s}_k(s - r).$$

When $k \le 0$, this is forward positive on all intervals I. When $k > 0$ this is forward positive on all intervals of length L satisfying $k \le (\pi/L)^2$.

(b) The Lagrange kernel for $y \mapsto y''' + ky'$ is

$$Q_k(s; r) = \begin{cases} \dfrac{1 - \mathbf{c}_k(s - r)}{k}, & k \ne 0; \\ \dfrac{(s - r)^2}{2}, & k = 0. \end{cases}$$

and this is forward positive on all intervals for all k.

Proof. A direct calculation using (3.7) shows these functions satisfy the conditions defining the Lagrange kernel and it is straightforward to check when they are forward positive. □

Lemma 3.9. *Let $\varkappa\colon [a, b] \to \mathbb{R}$ be continuous and let $u \in C^2([a, b])$ satisfy $u'' + \varkappa u = 0$.*

(a) *If* $u(a)=0$, $u'(a)\neq 0$, *and* $k_0\leq (\pi/(b-a))^2$, *then* $u\neq 0$ *on* (a,b).
(b) *If* $u(a)\neq 0$, $u'(a)=0$, *and* $k_0\leq (\pi/(2(b-a))^2$, *then* $u\neq 0$ *on* (a,b).

Proof. While (a) can be proven using the Sturm Comparison Theorem, [4, Theorem 1.1 Page 208], we give a short proof based on Theorem 3.7 which has the advantage of working for (b) as well. Without loss of generality we may assume $u'(a)=1$, in which case $u(s)>0$ for $s>a$ and s near a. Towards a contradiction assume u has a zero in (a,b) and let b^* be the smallest zero of u in (a,b). Then $u>0$ on (a,b^*). Let $\overline{u}(s):=\mathbf{s}_{k_0}(s-a)$. Then

$$\overline{u}''+k_0\overline{u}=0, \qquad \overline{u}(a)=0, \quad \overline{u}'(a)=1.$$

By Proposition 3.8 the Lagrange kernel for $\overline{u}\mapsto\overline{u}''+k_0\overline{u}$ is $\overline{K}(s;r)=P_{k_0}(s;r)=\mathbf{s}_{k_0}(s-r)$ and this is forward positive on all intervals (when $k_0\leq 0$) or on any interval whose length satisfies $k_0\leq(\pi/L)^2$ (when $k_0>0$). Therefore by Theorem 3.7 (with $n=2$ and $\ell=0$) we have $\overline{u}(s)=\mathbf{s}_{k_0}(s-a)\leq u(s)$ on $(0,b^*)$. As $u(b^*)=0$, this implies $\mathbf{s}_{k_0}(b^*-a)\leq 0$ so that $\mathbf{s}_{k_0}(s-a)=0$ would have a solution in $(a,b^*]$, which is not the case.

For (b) the same idea works, use 3.7 to compare u to $\overline{u}:=\mathbf{c}_{k_0}$. □

Theorem 3.10. *Let* I *be an interval and* $\varkappa\colon I\to\mathbb{R}$ *continuous. Then the Lagrange kernel for* $y\mapsto y'''+\varkappa y'$ *is forward positive in the following cases:*

(a) $\varkappa$ *is constant,*
(b) $\varkappa\leq 0$, *or*
(c) $\varkappa\leq k_1$ *for a positive constant* k_1 *with* $k_1\leq(\pi/L)^2$ *where* L *is the length of* I.

Proof. The case of $\varkappa$ being constant follows form Proposition 3.8.

The hypothesis of case (b) can be restated as $\varkappa\leq k_1:=0$. With this notation we have $\varkappa\leq k_1$ in both cases (b) and (c). Let let $y(s):=K(s;r)$, then $(y')''+\varkappa(y')=0$, $y'(a)=0$, $(y')'(a)=1$. Applying Lemma 3.9 to $u:=y'$ yields $y'>0$ on $I\cap(r,\infty)$ in both cases (b) and (c). Let $\overline{K}$ be the Lagrange kernel for the $\overline{y}\mapsto\overline{y}'''+k_1\overline{y}'$. Then $\overline{K}$ is forward positive by case (a). Theorem 3.7 (with $n=3$ and $\ell=1$) implies $y\geq\overline{y}$ on $I\cap(r,\infty)$. When $k_1=0$ we have $\overline{y}(s)=(s-r)^2/2>0$ and when $k_1>0$ we have $\overline{y}(s)=(1-\mathbf{c}_{k_1}(s-r))/k_1\geq 0$. As $y(s)=K(s;r)$ and r was any element of I we are done. □

Lemma 3.11. *Let $k \in \mathbb{R}$, then the solutions to the initial value problems*

$$\begin{aligned} \overline{x}_k''' + k\overline{x}_k' &= 0, \qquad \overline{x}_k(0) = 0, \quad \overline{x}_k'(0) = 1, \quad \overline{x}_k''(0) = 0 \\ \overline{y}_k''' + k\overline{y}_k' &= 0, \qquad \overline{y}_k(0) = 0, \quad \overline{y}_k'(0) = 0, \quad \overline{y}_k''(0) = 1 \end{aligned}$$

are

$$\overline{x}_k(s) = \mathbf{s}_k(s)$$
$$\overline{y}_k(s) = \begin{cases} \dfrac{1 - \mathbf{c}_k(s)}{k}, & k \neq 0 \\ \dfrac{s^2}{2}, & k = 0. \end{cases}$$

The curve $\overline{c}_k(s) := (\overline{x}_k(s), \overline{y}_k(s))$ has unit affine speed and constant affine curvature k. This parameterizes the connected component containing $(0,0)$ of the conic with equation

$$x^2 + ky^2 - 2y = 0.$$

Proof. Straightforward calculations using equations (3.7) and (3.8). □

Proposition 3.12. *Let $\varkappa$ be continuous on $[a,b]$ and let $x \in C^3([a,b])$ satisfy*

$$x''' + \varkappa x' = 0, \qquad x(a) = 0, \quad x'(a) = 1, \quad x''(a) = 0.$$

Assume for some constants k_0 and k_1 that $k_0 \leq \varkappa \leq k_1$ and $k_1 \leq (\pi/(2(b-a)))^2$. Then, with the notation of Lemma 3.11, the inequalities

$$\overline{x}_{k_1}(b-a) \leq x(b) \leq \overline{x}_{k_0}(b-a)$$

hold. If equality holds in the lower bound (respectively in the upper bound), then $\varkappa(s) = k_1$ and $x(s) = \overline{x}_{k_1}(s-a)$ (respectively $\varkappa = k_0$ and $x(s) = \overline{x}_{k_0}(s-a)$) on $[a,b]$.

Proposition 3.13. *Let $\varkappa$ be continuous on $[a,b]$ and let $y \in C^3([a,b])$ satisfy*

$$y''' + \varkappa y' = 0, \qquad y(a) = 0, \quad y'(a) = 0, \quad y''(a) = 1.$$

Assume for some constants k_0 and k_1 that $k_0 \leq \varkappa \leq k_1$ and $k_1 \leq (\pi/(b-a))^2$. Then, with the notation of Lemma 3.11, the inequalities

$$\overline{y}_{k_1}(b-a) \leq y(b) \leq \overline{y}_{k_0}(b-a)$$

hold. If equality holds in the lower bound (respectively in the upper bound), then $\varkappa(s) = k_1$ and $y(s) = \overline{y}_{k_1}(s-a)$ (respectively $\varkappa = k_0$ and $y(s) = \overline{y}_{k_0}(s-a)$) on $[a,b]$.

Proof of Propositions 3.12 and 3.13. We prove Proposition 3.12, the proof of Proposition 3.13 being similar. By Part (b) of Lemma 3.9 applied to the function $u = x'$ we see that $x' > 0$ on (a, b). Therefore the lower bound on x follows form Theorem 3.7 by comparing x to $\overline{x}(s) = \overline{x}_{k_1}(s)$ and the upper bound by comparing to $\overline{x}(s) := \overline{x}_{k_0}(s)$. Theorem 3.7 also covers the cases of equality. □

4. Comparisons for areas

Theorem 4.1. *Let D be a convex open set in $\mathbb{R}^2$ (which need not be bounded) with C^4 boundary. Let $p_0 \in \partial D$ and let $c\colon [0, L] \to \partial D$ have affine unit speed and $c(0) = p_0$ and let $\varkappa$ be the affine curvature of c. Let $\overline{\varkappa}\colon [0, L] \to \mathbb{R}$ be a continuous function so that the operator $\overline{y} \mapsto \overline{y}''' + \overline{\varkappa}\,\overline{y}'$ has forward positive Lagrange kernel and define $\overline{A}\colon [0, L] \to \mathbb{R}$ by the initial value problem*

$$\overline{A}''' + \overline{\varkappa}\overline{A}' = \frac{1}{2}, \qquad \overline{A}(0) = \overline{A}'(0) = \overline{A}''(0) = 0$$

Then, with $A_{c,0,c(0)}$ as in (2.1),

$$\varkappa(s) \leq \overline{\varkappa}(s) \text{ on } [0, L] \quad \text{implies} \quad A_{c,0,c(0)}(L) \geq \overline{A}(L)$$
$$\varkappa(s) \geq \overline{\varkappa}(s) \text{ on } [0, L] \quad \text{implies} \quad A_{c,0,c(0)}(L) \leq \overline{A}(L)$$

In either of these cases if equality holds, then $\varkappa \equiv \overline{\varkappa}$ on $[0, L]$.

Proof. To simplify notation let $A = A_{c,0,c(0)}$. The by Theorem 2.2 A satisfies the differential equation $A''' + \varkappa A = 1/2$ and $p_0 = c(0)$ the formulas (2.1) and (2.2) imply $A(0) = A'(0) = A''(0) = 0$. From Equation 2.2, $A'(s) = \frac{1}{2}(c(s) - c(0)) \wedge c'(s)$. This implies, see Figure 3, $A'(s) > 0$ all s with $c(s) \neq p_0$. Therefore the result follows from Theorem 3.7. □

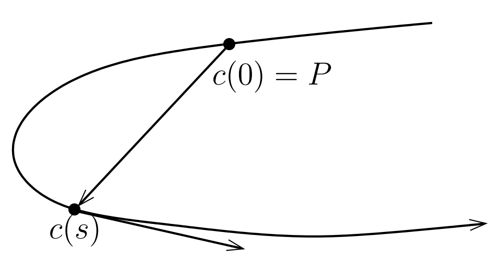

Figure 3. The vectors $c(s) - c(0)$ and $c'(s)$ form a right handed (i.e. positive) basis of $\mathbb{R}^2$.

Let $k \in \mathbb{R}$ and set

$$\overline{A}_k(s) = \begin{cases} \dfrac{s - \mathbf{s}_k(s)}{2k}, & k \neq 0; \\ \dfrac{s^3}{12}, & k = 0. \end{cases} \tag{4.1}$$

Using the definition of $\mathbf{s}_k$ it is not hard to check

$$\overline{A}_k'''(s) + k\overline{A}_k'(s) = \frac{1}{2} \qquad \overline{A}_k(0) = \overline{A}_k'(0) = \overline{A}_k''(0) = 0.$$

With the notation of Theorem 4.1, $A_k = A_{c,0,c(0)}$ where c is a curve with constant affine curvature k.

Corollary 4.2. *With notation as in Theorem 4.1 and Equation 4.1 if $k_0 \leq \varkappa \leq k_1$ for some constants k_0 and k_1, then*

$$\overline{A}_{k_1}(L) \leq A_{c,0,c(0)}(s) \leq \overline{A}_{k_0}(L)$$

on the interval $[0, L]$. If $A_{c,0,c(0)}(L) = \overline{A}_{k_0}(L)$ (respectively $A_{c,0,c(0)}(L) = \overline{A}_{k_1}(L)$) then c has constant curvature k_0 (respectively k_1) on the interval $[0, L]$. □

Proposition 4.3. *Let $\mathcal{C}$ be convex and have an affine curvature bound $\varkappa \geq k_0$. Then any triangle $\triangle p_1p_2p_3$ with vertices on $\mathcal{C}$ satisfies*

$$\operatorname{Area}(\triangle p_1p_2p_3) < \overline{A}_{k_0}(\Lambda(\mathcal{C})) \tag{4.2}$$

where $\overline{A}_{k_0}$ as in Equation (4.1) and $\Lambda(\mathcal{C})$ is the affine length of $\mathcal{C}$.

Proof. Let k_1 be any upper bound for the affine curvature of c. Then the result follows from Corollary 4.2. See Figure 4 □

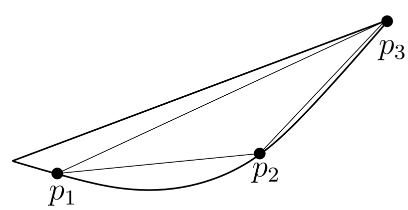

Figure 4. The area of $\triangle p_1p_2p_3$ is less than the area of the region bounded by $\mathcal{C}$ and the segment between the endpoints of $\mathcal{C}$ and this region has area at most $\overline{A}_{k_0}(\Lambda(\mathcal{C}))$ by Corollary 4.2.

Remark 4.4. The estimate in Proposition 4.3 is close to sharp in the sense that for a fixed affine length $\Lambda(\mathcal{C})$ as $k_0 \to -\infty$ the ratio of the two sides of inequality 4.2 goes to 1. To see this let $L > 0$ and $\mathcal{C}$ be the curve parameterized by $\overline{c}_{k_0} \colon [-L, L] \to \mathbb{R}^2$ be the curve of Lemma 3.11.

Then, as $\overline{c}_{k_0}$ has affine unit speed, $\Lambda(\mathcal{C}) = 2L$. Let p_1, p_2, and p_3 be the as in Figure 5. Then a calculation using that $\mathbf{s}_{k_0}(2L) = 2\mathbf{c}_{k_0}(L)\mathbf{s}_{k_0}(L)$ (which follows from the addition formula (3.10))

$$\begin{aligned}\frac{\operatorname{Area}(\triangle p_1p_2p_2)}{\overline{A}_{k_0}(\Lambda(\mathcal{C}))} - 1 &= -\frac{\mathbf{s}_{k_0}(L) - L}{\mathbf{s}_{k_0}(L)\mathbf{c}_{k_0}(L) - L} \\ &= -\frac{\sinh(\sqrt{|k_0|}L) - \sqrt{|k_0|}L}{\sinh(\sqrt{|k_0|}L)\cosh(\sqrt{|k_0|}L) - \sqrt{|k_0|}L} \\ &\sim \frac{-e^{-\sqrt{|k_0|}L}}{2} = \frac{-e^{-\sqrt{|k_0|}\Lambda(\mathcal{C})/2}}{2}.\end{aligned}$$

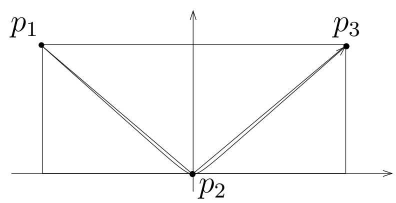

Figure 5. Here $p_1 = (\overline{x}_{k_0}(-L), \overline{y}_{k_0}(-L))$, $p_2 = (0,0)$, and $p_3 = (\overline{x}_{k_0}(L), \overline{y}_{k_0}(L))$. Then $\operatorname{Area}(\triangle p_1p_2p_3) = \overline{x}_{k_0}(L)\overline{y}_{k_0}(L)$. The curve is the hyperbola with equation $x^2 + k_0y^2 - 2y = 0$.

5. Adapted coordinates and geometric bounds.

Definition 5.1. Let $\mathcal{C}$ be a C^3 embedded convex curve and $p_0 \in \mathcal{C}$. Let $I_{\mathcal{C},p_0} \subseteq \mathbb{R}$ be the unique interval so that there is an affine unit speed parameterization $c_{\mathcal{C},p_0} \colon I_{\mathcal{C},p_0} \to \mathcal{C}$ with $c_{\mathcal{C},p_0}(0) = p_0$ that parameterizes all of $\mathcal{C}$. This is the ***standard parameterization*** of $\mathcal{C}$ at p_0. Then the ***affine adapted coordinates*** to $\mathcal{C}$ at p_0 are the linear coordinates ξ, η on $\mathbb{R}^n$ centered at p_0 with

$$\left.\frac{\partial}{\partial \xi}\right|_{p_0} = \mathbf{t}_{\mathcal{C}}(p_0) = c'_{\mathcal{C},p_0}(0), \qquad \left.\frac{\partial}{\partial \eta}\right|_{p_0} = \mathbf{n}_{\mathcal{C}}(p_0) = c''_{\mathcal{C},p_0}(0).$$

The ***graphing parameter set*** is the maximal interval $I^*_{\mathcal{C},p_0} \subseteq I_{\mathcal{C},p_0}$ so that the image of restriction $c_{\mathcal{C},p_0}\big|_{I^*_{\mathcal{C},p_0}} : I^*_{\mathcal{C},p_0} \to \mathcal{C}$ is a graph in the adapted coordinates ξ, η. The ***graphing interval***, $I^{**}_{\mathcal{C},p_0}$, is the set $I^{**}_{\mathcal{C},p_0} := \{\xi(c(s)) : s \in I^*_{\mathcal{C},p_0}\}$. See Figure 6.

Lemma 5.2. *Let ξ, η be affine adapted coordinates to $\mathcal{C}$ at p_0 and let $x, y \colon I_{\mathcal{C},p_0} \to \mathbb{R}$ be the coordinates of c in this coordinate system, that*

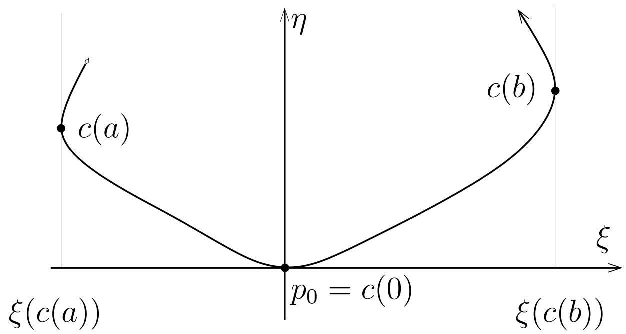

Figure 6. Here $c := c_{\mathcal{C},p_0}$ is the *standard parameterization* of $\mathcal{C}$ at p_0. The affine adapted coordinates centered at $p_0 = c(0)$ have the ξ-axis tangent to $\mathcal{C}$ at p_0 and the η-axis is in the direction of the affine normal $c''(0) = \mathbf{n}_{\mathcal{C}}(0)$. By local convexity the curve is locally the graph of a convex function. Let $[a,b]$ be the maximal subinterval of $I_{\mathcal{C},p_0}$ with $0 \in [a,b]$ so that the restriction $c\big|_{[a,b]}$ is is a graph. This is $I^*_{\mathcal{C},p_0}$, the *graphing parameter set.* The interval $[\xi(c(a)), \eta(c(b))]$ on the ξ-axis which is the domain of the graphing function is the *graphing interval* $I^{**}_{\mathcal{C},p_0}$. Note that the endpoints of $I^{**}_{\mathcal{C},p_0}$ are (when not an endpoint of $I_{\mathcal{C},p_0}$) are the points where the tangent $\mathcal{C}$ become vertical, that is if $x(s) = \xi(c(s))$, the points where $x'(s) = 0$.

is $x(s) := \xi(c(s))$ and $y(s) := \eta(c(s))$. Then x and y satisfy the initial value problems

$$\begin{aligned} &x''' + \varkappa x = 0, \qquad x(0) = 0 \qquad x'(0) = 1 \qquad x''(0) = 0 \\ &y''' + \varkappa y = 0, \qquad y(0) = 0 \qquad y'(0) = 0 \qquad y''(0) = 1 \end{aligned}$$

where $\varkappa$ is the affine curvature of $\mathcal{C}$.

Proof. This follows easily from the equation $c''' + \varkappa c' = 0$ and the definition of the adapted coordinates. □

Lemma 5.3. *Let $p_0 \in \mathcal{C}$. With the notation of Lemma 5.2 the connected component of* 0 *in $\{s \in I_{\mathcal{C},p_0} : x'(s) > 0\}$ is contained in the graphing parameter set, $I^*_{\mathcal{C},p_0}$.*

Proof. Let I_0 be connected component of 0 in $\{s \in I_{\mathcal{C},p_0} : x'(s) > 0\}$ and set $J := \{x(s) : s \in I_0\}$. Then, as $x'(s) > 0$ for I_0, the map $s \mapsto x(s)$ is a diffeomorphism between I_0 and J. Define $f\colon J \to \mathbb{R}$ by

$$f(x(s)) := \int_0^s y'(t)\,dt. \tag{5.1}$$

Taking the derivative of this gives $\frac{d}{ds}\left(f(x(s)) - y(s)\right) = 0$. Therefore $f(x(s)) - y(s) = C$ for some constant C. As $x(0) = y(0) = f(0) = 0$ we have $C = 0$, which implies $y(s) = f(x(s))$ for $s \in I_0$. Therefore I_0 is contained in the graphing parameter set about p_0. □

In the definition of the affine adapted coordinates it is assumed the curve $\mathcal{C}$ is of differentiability class C^3. In general the standard parameterization will only be C^2. If the standard parameterization is C^3 which is what is required for the affine curvature to be defined, there is a gain in the regularity of $\mathcal{C}$: it will be a C^4 immersed submanifold of $\mathbb{R}^2$ as we now show.

Proposition 5.4. *With notation as in Lemma 5.3 if $\varkappa$ is C^k for $k \geq 0$, then the function f is C^{k+4}. Thus the curve $\mathcal{C}$ is a C^{k+4} immersed submanifold of $\mathbb{R}^2$.*

Proof. As $y(s) = f(x(s))$ and x and y are C^3 it follows f is C^3. Taking three derivatives of (5.1) gives

$$\begin{aligned} y'(s) &= f'(x(s))x'(s) \\ y''(s) &= f''(x(s))x'(s)^2 + f'(x(s))x''(s) \\ y'''(s) &= f'''(x(s))x'(s)^3 + 3f''(x(s))x'(s)x''(s) + f'(x(s))x'''(s). \end{aligned} \tag{5.2}$$

Using $x'''(s) = -\varkappa(s)x'(s)$, $y'''(s) = -\varkappa(s)y'(s)$ and $y'(s) = f'(x(s))x'(s)$ in (5.2) gives

$$0 = f'''(x(s))x'(s)^3 + 3f''(x(s))x'(s)x''(s)$$

The function x' does not vanish on the interior of the graphing parametrization set so we can divide by $x'(s)^3$ to get

$$f'''(x(s)) = -\frac{3f''(x(s))x''(s)}{x'(s)^2}.$$

As x is C^3, and $x' \neq 0$ it has a C^3 inverse g, that there is a C^3 function g with $g(x(s)) = s$. Therefore

$$f'''(x) = -\frac{3f''(x)x''(g(x))}{x'(g(x))^2}$$

which shows f''' is C^1 and therefore f is C^4. For a non-parametric curve of class C^4 given as a graph $y = f(x)$ the affine curvature is (cf. [1, Page 14, eqn. (83)])

$$\varkappa(x) = -\frac{1}{2}\left(\frac{1}{(f''(x)^{\frac{2}{3}}}\right)'' = \frac{f''''(x)}{2f''(x)^{\frac{5}{3}}} - \frac{5f'''(x)^2}{9f''(x)^{\frac{8}{3}}}$$

which can be rewritten in the form

$$f''''(x) = 2\kappa(x) f''(x)^{\frac{5}{3}} + \frac{10}{9}\frac{f'''(x)^2}{f''(x)}$$

By standard regularity theorems for ordinary differential equations if $\varkappa$ is C^k then f is C^{k+4}. Or one can just take repeated derivatives of this equation and use induction to get the result. □

Theorem 5.5. *Let $\mathcal{C}$ be C^4 and let $p_0 \in \mathcal{C}$. Assume $(-L, L) \subseteq I_{\mathcal{C},p_0}$ for some $L > 0$. Also assume for some constants k_0 and k_1 the affine curvature of $\mathcal{C}$ satisfies the bounds $k_0 \le \varkappa(s) \le k_1$ for $-L \le s \le L$ and $k_1 \le (\pi/2L)^2$. Let ξ, η be affine adapted coordinates at p_0 let $x(s) = \xi(c(s))$ and $y(s) = \eta(c(s))$. Then $(-L, L)$ is contained in the graphing parameter set of $\mathcal{C}$ about p_0. With the notation of Lemma 3.11 the inequalities*

$$\overline{x}_{k_1}(|s|) \le |x(s)| \le \overline{x}_{k_0}(|s|) \tag{5.3}$$

$$\overline{y}_{k_1}(s) \le y(s) \le \overline{y}_{k_0}(s) \tag{5.4}$$

hold on the interval $(-L, L)$. Letting $R := \mathbf{s}_{k_1}(L)$, the graphing interval of $\mathcal{C}$ at s_0 contains $(-R, R)$. If $s_1 \in (-L, L)$ with $s_1 \ne 0$ and equality holds in either of the lower bounds of (5.3) *or* (5.4) *then the restriction of c to the interval of points between* 0 *and s_1 is a curve of constant curvature k_1. Likewise if equality holds in either of the upper bounds of* (5.3) *or* (5.4) *then the restriction of c to the interval of points between* 0 *and s_1 is a curve of constant curvature k_0.*

Proof. Applying Lemma 3.9 to the function $u = x'$ on $[0, L)$ yields $x' > 0$ on $[0, L)$. Applying the same lemma to $u(s) = x'(-s)$ on $[0, L)$ (which satisfies $u''(s) + \varkappa(-s)u(s) = 0$) implies $x' \ne 0$ the interval $(-L, L)$. Therefore $x' > 0$ on $(-L, L)$, thus by Lemma 5.3 the interval $(-L, L)$ is contained in the graphing parameter set of $\mathcal{C}$ at p_0. On the interval $[0, L)$ the inequalities 5.3 and 5.4 on the interval $[0, L)$ follow from Propositions 3.12 and 3.13.

To get the inequalities for $s \in (-L, 0]$ let $\widetilde{x}, \widetilde{y}\colon [0, -L) \to \mathbb{R}$ be given by

$$\widetilde{x}(s) = -x(-s), \qquad \widetilde{y}(s) = y(-s).$$

These satisfy

$$\widetilde{x}'''(s) + \varkappa(-s)\widetilde{x}'(s) = 0, \qquad \widetilde{x}(0) = 0, \quad \widetilde{x}'(0) = 1, \quad \widetilde{x}''(0) = 0,$$
$$\widetilde{y}'''(s) + \varkappa(-s)\widetilde{y}'(s) = 0, \qquad \widetilde{y}(0) = 0, \quad \widetilde{y}'(0) = 0, \quad \widetilde{y}''(0) = 1.$$

Again using Propositions 3.12 and 3.13 along with the definitions of $\widetilde{x}$ and $\widetilde{y}$ gives

$$\begin{aligned} \overline{x}_{k_1}(s) &\leq -x(-s) \leq \overline{x}_{k_0}(s) \\ \overline{y}_{k_1}(s) &\leq y(-s) \leq \overline{y}_{k_0}(s) \end{aligned}$$

on $[0, L)$. Replacing s by $-s$ and using $\overline{x}_{k_j}(-s) = -\overline{x}_{k_j}(s)$ and $\overline{y}_{k_j}(-s) = \overline{y}_{k_j}(s)$ shows the inequalities (5.3) and (5.4) also hold on $(-L, 0]$. The bound (5.3) implies the graphing interval contains $(-R, R)$.

The statements about when equality holds follow from the equality cases in Propositions 3.12 and 3.13. $\square$

Corollary 5.6. *Let $\mathcal{C}$ have the affine curvature bound $\varkappa \leq 0$ and assume there is an affine unit speed parameterization $c \colon \mathbb{R} \to \mathcal{C}$ defined on all of $\mathbb{R}$. Then c is bijective and for any point $p_0 \in \mathcal{C}$ the curve is globally a graph $\eta = f(\xi)$ in the affine adapted coordinates at p_0.*

Proof. Let $p_0 \in \mathcal{C}$. Without loss of generality we can assume $c(0) = p_0$. Let $L > 0$. In Theorem 5.5 let $k_0 = \min\{\varkappa(s) : s \in [-L, L]\}$ and $k_1 = 0$ to see the graphing parameter set contains $(-L, L)$ and the graphing interval contains $(-s_{k_1}(L), s_{k_1}(L)) = (-L, L)$ (as $\mathbf{s}_{k_1}(s) = \mathbf{s}_0(s) = s$). Letting $L \to \infty$ finishes the proof. $\square$

Example 5.7. To see that an upper bound on the affine curvature is necessary in this corollary note for any $k > 0$, let $\mathcal{C}$ be the circle with equation $x^2 + y^2 = k^{-3/4}$. Then $c \colon \mathbb{R} \to \mathcal{C}$ given by

$$c(s) = \big(k^{-3/4}\cos(k^{1/2}s), k^{-3/4}\sin(k^{1/2}s)\big)$$

is unit affine speed and $\mathcal{C}$ has constant curvature k, but c is not injective and $\mathcal{C}$ is not globally a graph in any coordinate system.

Theorem 5.8. *Let $\mathcal{C}$ have curvature bounds $k_0 \leq \varkappa \leq k_1$ with k_0, k_1 constants with $k_1 \leq (\pi/\Lambda(\mathcal{C}))^2$. Let p_1, p_2, p_3 be distant points on $\mathcal{C}$. Let $L = \Lambda(\mathcal{C})/2$. Then*

$$\operatorname{Area}(\triangle p_1p_2p_3) \leq \overline{x}_{k_0}(L)\overline{y}_{k_0}(L).$$

Equality holds if and only if $\mathcal{C}$ has constant curvature k_0 and, after maybe reordering, the points p_1, p_2, and p_3 are the initial point, midpoint, and endpoint of $\mathcal{C}$.

Proof. Let p_0 be the midpoint of $\mathcal{C}$ and construct the affine adapted coordinates ξ, η for $\mathcal{C}$ centered at p_0. Label the points p_1, p_2 and p_3 so that they are increasing order along $\mathcal{C}$. By Theorem 5.5 the curve lies inside the rectangle defined by $-\overline{x}_{k_0}(L) \leq \xi \leq \overline{x}_{k_0}(L)$ and $0 \leq \eta \leq \overline{y}_{k_0}(L)$ as shown in Figure 7. Any triangle in inside this rectangle has area at most half the area of the rectangle and therefore is at most $\overline{x}_{k_0}(L)\overline{y}_{k_0}(L)$. The only way that equality can hold is if the p_1 is the initial point of $\mathcal{C}$ and p_1 has coordinates $(-\overline{x}_{k_0}(L), \overline{y}_{k_0}(L))$, $p_2 = p_0$ is the midpoint of $\mathcal{C}$ and p_3 is the endpoint of $\mathcal{C}$ and has coordinates $(-\overline{x}_{k_0}(L), \overline{y}_{k_0}(L))$. This implies equality holds in the upper bounds of Theorem 5.5 and therefore $\mathcal{C}$ has constant curvature k_0. □

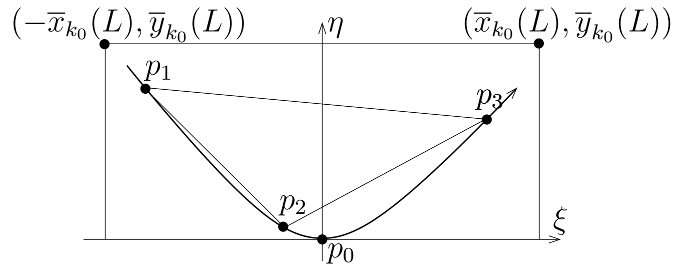

Figure 7. Let $L = \Lambda(\mathcal{C})/2$. Then Theorem 5.5 implies $\mathcal{C}$ is inside the pictured rectangle. Therefore $\triangle p_1p_2p_2$ has area at most half the area of this rectangle and equality only holds when p_1 is the upper left corner, p_3 is the upper right corner, and $p_1 = p_0$ is on the ξ-axis.

6. Bounds for the number of lattice points on curve.

Definition 6.1. Let $v_0, v_1, v_2 \in \mathbb{R}^2$ with v_1 and v_2 linearly independent. The ***lattice with origin*** v_0 ***and generated by*** v_1 ***and*** v_2 is the set

$$\mathcal{L}(v_0, v_1, v_2) := \{v_0 + mv_1 + nv_2 : m, n \in \mathbb{Z}\}.$$

The most basic invariant of a lattice is the area of its fundamental domain, $A_{\mathcal{L}}$. If $\mathcal{L} = \mathcal{L}(v_0, v_1, v_2)$ this is given by

$$A_{\mathcal{L}} = |v_1 \wedge v_2|.$$

Proposition 6.2. *If p_1, p_2, p_3 are three nonlinear points in $\mathcal{L} = \mathcal{L}(v_0, v_1, v_2)$, then the area of the triangle $\triangle p_1p_2p_3$ satisfies*

$$\operatorname{Area}(\triangle p_1p_2p_3) = \frac{\mathfrak{m}}{2} A_{\mathcal{L}}.$$

for some positive integer $\mathfrak{m}$ and therefore $\triangle p_1p_2p_3 \geq A_{\mathcal{L}}/2$.

Proof. If $p_j = v_0 + m_jv_1 + n_jv_2$, then

$$\begin{aligned}\operatorname{Area}(\triangle p_1p_2p_3) &= \frac{1}{2}|(p_2 - p_1) \wedge (p_3 - p_1)| \\ &= \frac{1}{2}|(m_2 - m_1)(n_3 - n_1) - (n_2 - n_1)(m_3 - m_1)||v_1 \wedge v_2| \\ &= \frac{\mathfrak{m}}{2} A_{\mathcal{L}}\end{aligned}$$

where $\mathfrak{m} = |(m_2 - m_1)(n_3 - n_1) - (n_2 - n_1)(m_3 - m_1)|$ is a positive integer. $\square$

As an example to motivate the following definition consider the ellipse defined by $ax^2 + bxy + cy^2 = R$ where a, c, R are odd integers and b is an even integer. Let $\mathcal{L} = \mathbb{Z}^2$ be the standard integral lattice. If $p = (x, y) \in \mathcal{L} \cap \mathcal{C}$ then reducing $ax^2 + bxy + cy^2 = R^2$ modulo 2 (and using $x^2 \equiv x \bmod 2$) gives $x + y \equiv 1 \bmod 2$. Whence $p \equiv (1, 0)$ or $p \equiv (0, 1)$ modulo 2. Thus if $p_j = (x_j, y_j) \in \mathcal{L} \cap \mathcal{C}$ for $j = 1, 2, 3$, then for at least at least one pair from $\{p_1, p_2, p_3\}$, say p_1, p_2 we have $p_1 \equiv p_2 \bmod 2$. This implies $(p_2 - p_1) \wedge (p_3 - p_1)$ is even and therefore $\operatorname{Area}(\triangle p_1 p_2 p_3) = \frac{1}{2}|(p_2 - p_1) \wedge (p_3 - p_1)|$ is an integer and whence is twice as large as $1/2$ which the minimum area of a general triangle with vertices in $\mathbb{Z}^2$.

Definition 6.3. Let $\mathcal{L}$ be a lattice and $\mathcal{C}$ a curve. Then $\mathfrak{m}(\mathcal{C}, \mathcal{L})$ is the largest positive integer so that

$$\operatorname{Area}(\triangle p_1 p_2 p_2) \geq \frac{\mathfrak{m}(\mathcal{C}, \mathcal{L})}{2} A_{\mathcal{L}}$$

for all distinct points $p_1, p_2, p_3 \in \mathcal{L} \cap \mathcal{C}$.

Defining and using the integer $\mathfrak{m}(\mathcal{C}, \mathcal{L})$ to improve lattice point estimates is an abstraction of an idea in the paper [12] of Ramana where a related integer, m_{ad} is defined for the integer lattice and integral conics of the form $ax^2 + dy^2 = R$ and the lattice $\mathbb{Z}^2$.

Definition 6.4. For each $k \in \mathbb{R}$ let $\overline{F}_k$ be the inverse of function $\overline{A}_k$. (The function $\overline{A}_k$ is strictly increasing and therefore this inverse exists.)

Theorem 6.5. *Let $\mathcal{C}$ have be convex, C^4, and with affine length $\Lambda(\mathcal{C})$ and a lower bound $\varkappa \geq k_0$ on its affine curvature. Let $\mathfrak{m} = \mathfrak{m}(\mathcal{C}, \mathcal{L})$. If*

$$\overline{A}_{k_0}(\Lambda(\mathcal{C})) \leq \frac{\mathfrak{m} A_{\mathcal{L}}}{2} \tag{6.1}$$

then, there are at most two points of $\mathcal{L}$ on $\mathcal{C}$.

Proof. If there are three or more points of $\mathcal{L}$ on c, let p_1, p_2, and p_3 be three of them. By Propositions 6.2 and 4.3 and the definition of $\mathfrak{m} = \mathfrak{m}(\mathcal{C}, \mathcal{L})$ this implies

$$\frac{\mathfrak{m} A_{\mathcal{L}}}{2} \leq \operatorname{Area}(\triangle p_1 p_2 p_3) < \overline{A}_{k_0}(\Lambda(\mathcal{C}))$$

which contradicts (6.1). $\square$

Theorem 6.6. *Let $\mathcal{L}$ be a lattice, $\mathcal{C}$ a C^4 curve with a lower curvature bound $\varkappa \geq k_0$ on its affine curvature, and $\mathfrak{m} = \mathfrak{m}(\mathcal{C}, \mathcal{L})$ as in definition 6.3. Assume that $\mathcal{C}$ is convex, or more generally that any sub-arc of affine length at most $F_{k_0}(A_{\mathcal{L}}/2)$ is convex. Then*

$$\#(\mathcal{L} \cap \mathcal{C}) \leq 2 \left\lceil \frac{\Lambda(\mathcal{C})}{F_{k_0}(\mathfrak{m} A_{\mathcal{L}}/2)} \right\rceil .$$

Proof. By Theorem 6.5 any sub-arc of $\mathcal{C}$ with affine length at most $F_{k_0}(\mathfrak{m} A_{\mathcal{L}}/2)$ contains at most two points of $\mathcal{L}$. Let $m = \lceil \Lambda(\mathcal{C})/F_{k_0}(\mathfrak{m} A_{\mathcal{L}}/2) \rceil$. By dividing $\mathcal{C}$ into m sub-arcs of affine length L/m we cover c with m sub-arcs with affine length $\leq F_{k_0}(\mathfrak{m} A_{\mathcal{L}}/2)$. As each of these sub-arcs contains at most two points of of $\mathcal{L}$ the total number of points in $\mathcal{L} \cap \mathcal{C}$ is at most $2m$. □

Theorem 6.7. *Let $\mathcal{C}$ have affine curvature bounds $k_0 \leq \varkappa \leq k_1$ with $k_1 \leq (\pi/\Lambda(\mathcal{C}))^2$. Let $\mathcal{L}$ be a lattice and let $L = \Lambda(\mathcal{C})/2$ and $\mathfrak{m} = \mathfrak{m}(\mathcal{C}, \mathcal{L})$. If*

$$\overline{x}_{k_0}(L)\overline{y}_{k_0}(L) \leq \frac{\mathfrak{m} A_{\mathcal{L}}}{2}, \tag{6.2}$$

then

$$\#(\mathcal{L} \cap \mathcal{C}) \leq 3.$$

The equality $\#(\mathcal{L} \cap \mathcal{C}) = 3$ holds if and only if $\mathcal{C}$ has constant curvature k_0, equality holds in (6.2) *and the three points of $\mathcal{L}$ on $\mathcal{C}$ are the two endpoints of $\mathcal{C}$ along with the midpoint of $\mathcal{C}$.*

Proof. If there are more than two points of $\mathcal{L}$ on $\mathcal{C}$, let p_1, p_2, p_3 be three of them. By Theorem 5.5 these points are on a convex graph and therefore are not collinear. Thus by Proposition 6.2, Theorem 5.8, the definition of $\mathfrak{m}$ and the inequality (6.2)

$$\frac{\mathfrak{m} A_{\mathcal{L}}}{2} \leq \operatorname{Area}(\triangle p_1 p_2 p_3) \leq \overline{x}_{k_0}(L)\overline{y}_{k_0}(L) \leq \frac{\mathfrak{m} A_{\mathcal{L}}}{2}.$$

Therefore equality holds in Theorem 5.8, which happens if and only if these three points are the midpoint of $\mathcal{C}$ along with the endpoints of $\mathcal{C}$ and $\mathcal{C}$ has constant affine curvature k_0. This shows that any size three subset of $\mathcal{L} \cap \mathcal{C}$ consists of the endpoints and midpoint of $\mathcal{C}$ and thus $\mathcal{C}$ has at most three points. □

Lemma 6.8. *Let $k \in \mathbb{R}$ and define intervals I_k and J_k by*

$$I_k := \begin{cases} [0, \infty), & k \leq 0; \\ [0, \pi/2\sqrt{k}\,], & k > 0. \end{cases} \qquad J_k := \begin{cases} [0, \infty), & k \leq 0; \\ [0, 1/k^{3/2}], & k > 0. \end{cases}$$

and let H_k be defined on I_k by $H_k(s) = \overline{x}_k(s)\overline{y}_k(s)$. Then H_k is a homeomorphism between I_k and J_k.

Proof. It is elementary to check that each of $\overline{x}_k$ and $\overline{y}_k$ are strictly increasing on I_k, and therefore H_k is also strictly increasing. Also $\lim_{s\to\infty} H_k(s) = \infty$ when $k \le 0$ and $H_k(\pi/2\sqrt{k}) = 1/k^{3/2}$ when $k > 0$. This implies H_k is a bijective continuous map between the intervals and thus a homeomorphism. □

Definition 6.9. Let $G_k \colon J_k \to I_k$ be the inverse of the map H_k of Lemma 6.8

Theorem 6.10. *Let $\mathcal{C}$ be a curve with affine curvature bounds $k_0 \le \varkappa \le k_1$. Let $\mathcal{L}$ be a lattice, $\mathfrak{m} = \mathfrak{m}(\mathcal{C}, \mathcal{L})$, and set $L = G_{k_0}(\mathfrak{m}A_{\mathcal{L}}/2)$ and $m = \lfloor \Lambda(\mathcal{C})/(2L)\rfloor$. If $k_1 > 0$ also assume $k_1 \le (\pi/(2L))^2$. Then*

$$\#(\mathcal{L} \cap \mathcal{C}) \le 2m + 2.$$

Proof. and let $c \colon [0, \Lambda(\mathcal{C})] \to \mathbb{R}^2$ be an affine unit speed parameterization of $\mathcal{C}$. For $j = 1, 2, \dots, m$ define sub-arcs of $\mathcal{C}$ by

$$\begin{aligned} \mathcal{C}_j &:= \{c(s) : 2(j-1)L \le s < 2jL\} \\ \mathcal{C}_j^* &:= \{c(s) : 2(j-1)L \le s \le 2jL\} \\ \mathcal{C}_{m+1} &:= \{c(s) : 2mL \le s \le \Lambda(\mathcal{C})\}. \end{aligned}$$

Then $\mathcal{C}_j$ is just $\mathcal{C}_j^*$ with its right endpoint removed. The affine length of $\mathcal{C}_j^*$ is $2L = G_{k_0}(\mathfrak{m}A_{\mathcal{L}}/2)$ and by the definition G_{k_0} we have $\overline{x}_{k_0}(L)\overline{y}_{k_0}(L) = \mathfrak{m}A_{\mathcal{L}}/2$. Therefore by Theorem 6.7 the arc $\mathcal{C}_j^*$ contains at most 3 points of $\mathcal{L}$ and if it does contain 3 points, then two of these points are endpoints of $\mathcal{C}_j^*$. Thus $\mathcal{C}_j$ contains at most two points of $\mathcal{L}$. The arc $\mathcal{C}_{m+1}$ has affine length less than $2L$, using Theorem 6.7 again, it contains at most two points of $\mathcal{L}$. As $\mathcal{C} = \mathcal{C}_1 \cup \mathcal{C}_2 \cup \cdots \cup \mathcal{C}_{m+1}$ this implies $\mathcal{C} \cap \mathcal{L}$ contains at most $2(m+1)$ points. □

There is a rigidity version of this result.

Theorem 6.11. *Let $\mathcal{C}$ satisfy the hypothesis of Theorem 6.10 with the extra assumption that*

$$m = \frac{\Lambda(\mathcal{C})}{2L}$$

is an integer. Then if $\mathcal{C}$ is open (that is not the boundary of a bounded convex domain) then

$$\#(\mathcal{L} \cap \mathcal{C}) \le 2m + 1.$$

Equality holds if and only if $\mathcal{C}$ has constant curvature k_0 and the points of $\mathcal{L} \cap \mathcal{C}$ are evenly spaced along $\mathcal{C}$ with respect to affine arc length at a distance of $L = G_{k_0}(\mathfrak{m}A_{\mathcal{L}}/2)$ between consecutive points. In particular the endpoints of $\mathcal{C}$ are in $\mathcal{L}$.

Proof. That m is an integer implies in the proof of Theorem 6.10 that $\mathcal{C}_{m+1}$ is just the one point set $\{c(\Lambda(\mathcal{C}))\}$. Thus each $\mathcal{C}_j$ contains at most two points of $\mathcal{L}$ for $j = 1, 2, \ldots, m$ and $\mathcal{C}_{m+1}$ contains at most one point of $\mathcal{L}$. Thus $\mathcal{C} = \mathcal{C}_1 \cup \mathcal{C}_2 \cup \cdots \cup \mathcal{C}_{m+1}$ contains at most $2m+1$ points of $\mathcal{L}$.

We prove that when equality holds that $\mathcal{C}$ has constant curvature and that the points of $\mathcal{C} \cap \mathcal{L}$ are evenly spaced by induction on m. If $m = 1$, then the result follows from Theorem 6.7 because $\overline{x}_{k_0}(L)\overline{y}_{k_0}(L) = \mathfrak{m}A_{\mathcal{L}}/2$.

Assume the result holds for m and let $\mathcal{C}$ be a curve with $\Lambda(\mathcal{C}) = (m+1)G_{k_0}(\mathfrak{m}A_{\mathcal{L}}/2)$ and $\#(\mathcal{L}\cap\mathcal{C}) = 2m+3$. Let $c\colon [0, \Lambda(\mathcal{C})] \to \mathcal{C}$ be an affine unit speed parameterization of $\mathcal{C}$ and let $\mathcal{C}' := \{c(s) : 0 \le s \le 2L\}$ and $\mathcal{C}'' := \{c(s) : 2L \le s \le (m+1)L = \Lambda(\mathcal{C})\}$. Then $\mathcal{C}'$ has at most 3 points and $\mathcal{C}''$ has at most $2m+1$ points. The intersection $\mathcal{C}' \cap \mathcal{C}''$ only has the one point $c(2L)$. If this point is not in $\mathcal{L}$, then by the induction hypothesis $\#(\mathcal{L} \cap \mathcal{C}') \le 2$ and $\#(\mathcal{L} \cap \mathcal{C}'') \le 2m$ and the set $\mathcal{L} \cap \mathcal{C}'$ and $\mathcal{L} \cap \mathcal{C}''$ have no point in common. Thus $\#(\mathcal{L} \cap \mathcal{C}) = \#(\mathcal{L} \cap (\mathcal{C}' \cup \mathcal{C}'')) \le 2 + 2m$ contradicting that $\#(\mathcal{L} \cap \mathcal{C}) = 2m + 3$. Therefore $\mathcal{L} \cap \mathcal{C}'$ and $\mathcal{L} \cap \mathcal{C}''$ have one point in common, whence

$$\#(\mathcal{L} \cap \mathcal{C}) = \#(\mathcal{L} \cap \mathcal{C}') + \#(\mathcal{L} \cap \mathcal{C}'') - 1 \le 3 + 2m + 1 - 1 = 2m + 3.$$

Thus the assumption $\#(\mathcal{L} \cap \mathcal{C}) = 2m + 3$ implies $\#(\mathcal{L} \cap \mathcal{C}') = 3$ and $\#(\mathcal{L}\cap\mathcal{C}'') = 2m+1$. Therefore the induction hypothesis implies $\mathcal{C}'$ and $\mathcal{C}''$, and therefore $\mathcal{C}$, have constant affine curvature k_0 and the points are equality spaced at a distance of L between consecutive points. □

7. Examples.

In this section we give examples to show our theorems bounding the number of lattice points on a curve are sharp. To simplify things in all of these examples will have $\mathfrak{m}(\mathcal{L}, \mathcal{C}) = 1$.

7.1. **Examples when $k_0 = 0$.** In this case the function $H_{k_0} = H_0$ of Lemma 6.8 is given by $H_0(s) = s^3/2$ and therefore its inverse (cf. Definition 6.9) is

$$G_0(s) = (2s)^{1/3}.$$

Let $\mathcal{L} = \mathcal{L}(v_0, v_1, v_2)$ be a lattice. By possibly replacing v_2 by $-v_2$ we may assume $v_1 \wedge v_2 > 0$. Let $\alpha = (v_1 \wedge v_2)^{-1/3}$ and let $\mathcal{C}$ be the parabola parameterized by

$$c(s) = v_0 + (\alpha s)v_2 + \frac{(\alpha s)(\alpha s + 1)}{2} v_2.$$

A bit of calculation shows

$$c'(s) \wedge c''(s) = \alpha^3 v_1 \wedge v_2 = 1.$$

Therefore c is an affine unit speed parameterization of $\mathcal{C}$. Let $s_j = j/\alpha$. Then

$$p_j := c(s_j) = v_0 + jv_1 + \frac{j(j+1)}{2} v_2 \in \mathcal{L}$$

and the affine distance between p_{j+1} and p_j is $s_{j-1} - s_j = 1/\alpha$. The L of Theorems 6.10 and 6.11 is given by

$$L = G_0(A_{\mathcal{L}}/2) = (A_{\mathcal{L}})^{1/3} = (v_1 \wedge v_2)^{1/3} = 1/\alpha,$$

which is the affine distance between p_j and p_{j+1} on $\mathcal{C}$.

Letting $k_0 = 0$ and $k_1 > 0$ with $k_1 \leq (\pi/2L)^2$ we then have that the restriction $\mathcal{C}\big|_{p_1}^{p_{2m+1}}$, that is the arc of $\mathcal{C}$ between p_1 and p_{2m+1}, gives an example where equality holds in Theorem 6.11 and thus also in Theorem 6.7. The curve $\mathcal{C}\big|_{p_0}^{p_{m+1}}$ is an example where equality holds in Theorem 6.10.

7.2. **Constructing closely spaced lattice points on conics.**

Lemma 7.1. *Let $\mathcal{L} = \mathcal{L}(v_0, v_1, v_2)$ and $\mathcal{L}' = \mathcal{L}(v_0', v_1', v_2')$ Assume $\mathcal{L}' \subseteq \mathcal{L}$ and $A_{\mathcal{L}} = A_{\mathcal{L}'}$. Then $\mathcal{L} = \mathcal{L}'$.*

Proof. As $\mathcal{L}' \subseteq \mathcal{L}$ we have $v_0' \in \mathcal{L}$ and therefore we can write $\mathcal{L}$ as $\mathcal{L} = \mathcal{L}(v_0', v_1, v_2)$. Then $\mathcal{L}' \subseteq \mathcal{L}$ implies there are integers a_{ij} so that

$$\begin{aligned} v_0' + v_1' &= v_0' + a_{11}v_1 + a_{12}v_2 \\ v_0' + v_2' &= v_0' + a_{21}v_1 + a_{22}v_2 \end{aligned}$$

The equality $A_{\mathcal{L}} = A_{\mathcal{L}'}$ implies $v_1 \wedge v_2 = \pm v_1' \wedge v_2'$. Therefore

$$\begin{aligned} v_1' \wedge v_2' &= (a_{11}v_1 + a_{12}v_2) \wedge (a_{21}v_1 + a_{22}v_2) \\ &= (a_{11}a_{22} - a_{12}a_{21}) v_1 \wedge v_2 \\ &= \pm(a_{11}a_{22} - a_{12}a_{21}) v_1' \wedge v_2' \end{aligned}$$

Thus $a_{11}a_{22} - a_{12}a_{21} = \pm 1$. Whence if $[b_{ij}] = [a_{ij}]^{-1}$ is the inverse of the matrix $[a_{ij}]$, then, using Cramers's rule for the inverse, we see the numbers b_{ij} are integers. Then

$$\begin{aligned} v_0' + v_1 &= v_0' + b_{11}v_1' + b_{12}v_2' \\ v_0' + v_2 &= v_0' + b_{21}v_1' + b_{22}v_2' \end{aligned}$$

This implies $\mathcal{L} \subseteq \mathcal{L}'$ and thus $\mathcal{L} = \mathcal{L}'$. □

Lemma 7.2. *Let $\mathcal{L}$ be a lattice and φ an affine motion of $\mathbb{R}^2$ so that for some points $p_0, p_1, p_2 \in \mathcal{L}$ we have $\varphi(p_0), \varphi(p_1), \varphi(p_2) \in \mathcal{L}$ and*

$$\text{Area}(\triangle\varphi(p_0)\varphi(p_1)\varphi(p_2)) = \frac{A_{\mathcal{L}}}{2.}$$

Then φ preserves the lattice $\mathcal{L}$.

Proof. As φ is an affine motion it preserves area. Therefore

$$\text{Area}(\triangle p_0p_1p_2) = \text{Area}(\triangle\varphi(p_1)\varphi(p_2)\varphi(p_3)) = A_{\mathcal{L}}/2.$$

As $p_0, p_1, p_2 \in \mathcal{L}$ we have $\mathcal{L}_1 := \mathcal{L}(p_0, p_1 - p_0, p_2 - p_0) \subseteq \mathcal{L}$. Then $\text{Area}(\triangle p_0p_1p_2) = A_{\mathcal{L}}/2$ implies $A_{\mathcal{L}_1} = A_{\mathcal{L}}$ and by Lemma 7.1 we have $\mathcal{L}_1 = \mathcal{L}$. The same argument shows $\mathcal{L} = \mathcal{L}(\varphi(p_0), \varphi(p_1) - \varphi(p_0), \varphi(p_2) - \varphi(p_0)) = \mathcal{L}$. Thus the image of $\mathcal{L}$ under φ is

$$\begin{aligned}\varphi[\mathcal{L}] &= \varphi[\mathcal{L}(p_0, p_1 - p_0, p_2 - p_0)] \\ &= \mathcal{L}(\varphi(p_0), \varphi(p_1) - \varphi(p_0), \varphi(p_2) - \varphi(p_0)) \\ &= \mathcal{L}\end{aligned}$$

as required. □

Proposition 7.3. *Let $\mathcal{C}$ be connected component of a conic with constant affine curvature k_0 and let $\mathcal{L}$ be a lattice. Assume there are distinct points let p_1, p_2, p_3, p_4 listed in increasing order with respect to the natural orientation on $\mathcal{C}$ with the affine distance between p_j and $p_{j+1} = L$ for some constant L and $j = 1, 2, 3$ and with*

$$\text{Area}(p_2, p_3, p_4) = \frac{A_{\mathcal{L}}}{2}.$$

Then there is a unique special affine motion φ which preserves both the curve $\mathcal{C}$ and the lattice $\mathcal{L}$ and with $\varphi(p_j) = p_{j+1}$ for $j = 1, 2, 3$. Also

(a) for all integers j we have $\varphi^j(p_1) \in \mathcal{L} \cap \mathcal{C}$ and for all j the affine distance between $p_j := \varphi^{j-1}(p_1)$ and $p_{j+1} := \varphi^j(p_1)$ is L.

(b) if k_0 is the curvature of $\mathcal{C}$ then k_0, L and $A_{\mathcal{L}}$ are related by

$$\frac{A_{\mathcal{L}}}{2} = H_{k_0}(L). \tag{7.1}$$

where H_{k_0} is as in Lemma 6.8.

Proof. Let $c_1 \colon \mathbb{R} \to \mathcal{C}$ be an affine unit speed parametrization of of $\mathcal{C}$ with $c_1(0) = p_1$. Then $p_j = c((j-1)L)$ for $j = 1, 2, 3, 4$. Let $c_2(s) = c_1(s + L)$. Then c_2 is an affine unit speed parameterization of $\mathcal{C}$. As $\mathcal{C}$ has constant affine curvature Theorem 2.1 gives us a unique affine motion φ with $c_2(s) = \varphi(c_1(s))$. Then $\varphi(c_1(s)) = c_1(s+L)$ for all s and thus φ preserves $\mathcal{C}$ and $\varphi(p_j) = p_{j+1}$ for $j = 1, 2, 3$. Also

$$\operatorname{Area}(\triangle \varphi(p_1)\varphi(p_2)\varphi(p_3)) = \operatorname{Area}(\triangle p_2 p_3 p_4) = \frac{A_{\mathcal{L}}}{2}.$$

Therefore, by Lemma 7.2, φ preserves $\mathcal{L}$. For all integers j we have $\varphi^j(p_1) = c_1(jL) \in \mathcal{L}$ and φ preserves affine distance, therefore the affine distance between p_j and p_{j+1} is L. Finally equation (7.1) holds by Theorem 6.7. $\square$

7.3. **Examples when $k_0 < 0$.** We first consider the case where the lattice is $\mathcal{L} = \mathbb{Z} \times \mathbb{Z}$ is the lattice of integers points in the plane. Let $\mathcal{C}_0$ be the connected component of the hyperbola with equation

$$x^2 - xy - y^2 = 1$$

which contains the point $(1, 0)$. Let

$$\alpha := 2^{-1/3} 5^{1/6}$$

Then a calculation shows that $c\colon \mathbb{R} \to \mathbb{R}^2$ given by

$$c(s) = \left(\cosh(\alpha s) - \frac{1}{\sqrt{5}} \sinh(\alpha s), -\frac{2}{\sqrt{5}} \sinh(\alpha s) \right)$$

is a affine unit speed parameterization of $\mathcal{C}_0$ with

$$c(0) = (1, 0).$$

It is not hard to see that

$$c'''(s) = \alpha^2 c'(s)$$

and therefore $\mathcal{C}$ has constant curvature $k_0 = -\alpha^2$. Let

$$L = \frac{1}{\alpha} \operatorname{arcsinh}(\sqrt{5}/2).$$

Then $\sinh(\alpha L) = \sqrt{5}/2$ and $\cosh(\alpha L) = \sqrt{(1 + (\sqrt{5}/2)^2)} = 3/2$. Therefore

$$c(L) = (1, -1).$$

To find more integral points on $\mathcal{C}$ let φ be the linear map $\varphi\colon \mathbb{R}^2 \to \mathbb{R}^2$ given by

$$\varphi(x, y) := (x - y, -x + 2y).$$

Then φ is an affine motion and preserves both $\mathcal{C}$ and $\mathcal{L}$. Define $p_1 := c(0) = (1, 0)$ and $p_2 := c(L) = (1, -1)$, then $\varphi(p_1) = p_2$. It follows that for all $j \in \mathbb{Z}$ the points[1]. $p_j := c((j-1)L) = \varphi^{j-1}(p_1)$ are in $\mathcal{L} \cap \mathcal{C}$. Then

[1]For $j > 0$ it is not hard to check if $f_0, f_1, f_2 \dots$ is the Fibonacci sequence defined by $f_{j+2} = f_{j+1} + f_j$, $f_0 = 0$, $f_1 = 1$, then for $j \geq 2$ the points are $p_j = (f_{2j-3}, -f_{2j-2})$

$p_2 = (2, -3)$ and $p_4 = (5, -8)$, then $\text{Area}(\triangle p_2 p_3 p_4) = 1/2 = A_{\mathcal{L}}/2$ (See Figure 8). Therefore Proposition 7.3 implies $H_{k_0}(L) = A_{\mathcal{L}}/2$, or what is the same thing $G_{k_0}(A_{\mathcal{L}}/2) = L$.

Example 7.4. If $\mathcal{C}_1 := \{c(s) : 0 \le s \le (2m+1)L\}$, then this gives an example where equality holds in Theorem 6.10. Letting $\mathcal{C}_2 := \{c(s) : L \le s \le (2m+1)L\}$ gives an example where equality holds in Theorem 6.11.

Example 7.5. The previous example can be transferred to other lattices. Let $\mathcal{L} = \mathcal{L}(v_0, v_1, v_2)$. We can assume $v_1 \wedge v_2 > 0$ (if not replace v_2 by $-v_2$). With α and L as in the previous example let

$$\beta = (v_1 \wedge v_2)^{1/3}$$

and

$$\widehat{c}(s) = v_0 + \left(\cosh(\alpha\beta s) - \frac{1}{\sqrt{5}}\sinh(\alpha\beta s)\right) v_1 - \left(\frac{2}{\sqrt{5}}\sinh(\alpha\beta s)\right) v_2.$$

Let $\widehat{L} = L/\beta$. Then the curve $\{\widehat{c}(s) : \widehat{L} \le s \le (2m+1)\widehat{L}\}$ is an example where equality holds in Theorem 6.11 and for the curve $\{\widehat{c}(s) : 0 \le s \le (2m+1)\widehat{L}\}$ equality holds in Theorem 6.10.

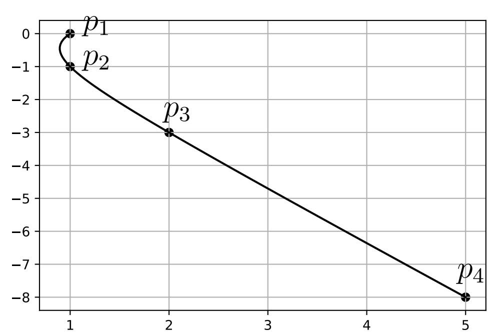

Figure 8. The branch of $x^2 - xy - y^2 = 1$ through $(1, 0)$ showing the lattice points on this curve with y coordinate satisfying $-8 \le y \le 0$. Here $p_1 = (1, 0)$, $p_2 = (1, -1)$, $p_3 = (2, -3)$ and $p_4 = (5, -8)$.

7.4. **Examples when $k_0 > 0$.** In looking for examples our results are sharp when the lower curvature bound, k_0, is positive we need to find evenly spaced points on a curve, $\mathcal{C}$, with constant curvature k_0. Such a curve is a ellipse and after an affine motion we may assume it is a circle centered at the origin with radius $r = k_0^{-3/2}$ and having unit affine speed parametrization

$$c(s) = (r\cos(k_0^{1/2}s), r\sin(k_0^{1/2}s)).$$

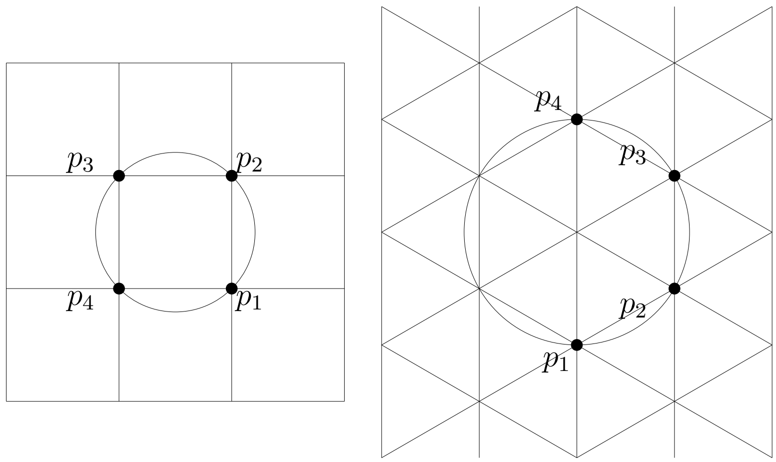

Figure 9. In the case of a curve of positive constant affine curvature up to an affine motion the only configurations where there are four points satisfying from the lattice on the curve that they are evenly spaced with respect to affine arc length and so that Area($\triangle p_1p_2p_2$) is half the area of a fundamental region of the lattice are shown here. (In the second figure note that a fundamental region for the lattice is not on of the triangles, but a parallelogram consisting of two of the triangles.)

Let $\mathcal{L}$ be a lattice so that there are four points $p_1, p_2, p_3, p_4 \in \mathcal{L} \cap \mathcal{C}$ that are equally spaced with respect to affine arc length and so that $H_{k_0}(L) = A_{\mathcal{L}}/2$ where L is the affine distance between p_j and p_{j+1}. If equality holds in Theorem 6.11 with $m \geq 2$ then four such points exist. Proposition 7.3 gives an affine motion φ that preserves both $\mathcal{L}$ and $\mathcal{C}$. As $\mathcal{C}$ is a circle centered at the origin this implies φ is a rotation about the origin. (The rotation being with respect to the Euclidean structure that makes $\mathcal{C}$ into a circle of radius r.) If $\mathcal{L} = \mathcal{L}(v_0, v_1, v_2)$ and $\varphi(v) = Mv$ (it is linear as it fixes the origin) then the matrix of M with respect to the basis v_1, v_2 has integer entries and therefore its trace is an integer. As M is a rotation, its matrix with respect to the standard basis is $\begin{bmatrix}\cos(\theta) & -\sin(\theta)\\ \sin(\theta) & \cos(\theta)\end{bmatrix}$ where θ is the angle of rotation. The trace of this is $2\cos(\theta)$ and therefore $2\cos(\theta)$ is an integer. This implies θ is either an integral multiple of either $\pi/3$ or $\pi/2$. The only lattices where we can get four points equally space along the circle and so that Area($\triangle p_2p_3p_4$) is half the area of a fundamental region of the lattice are shown in Figure 9. Therefore Theorems 6.10 and 6.11 can only be sharp when $\#(\mathcal{L}\cap\mathcal{C})$ is small to be precise $\#(\mathcal{L}\cap\mathcal{C})\leq 6$.

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