MAPS THAT MUST BE AFFINE OR CONJUGATE AFFINE: A PROBLEM OF VLADIMIR ARNOLD

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Abstract. A k-flat in a vector space is a k-dimensional affine subspace. Our basic result is that an injection $T\colon \mathbb{C}^n \to \mathbb{C}^n$ that for some $k \in \{1, 2, , \dots, n-1\}$ T maps all k-flats to flats of $\mathbb{C}^n$ and is either continuous at a point or Lebesgue measurable, is either an affine map or a conjugate affine map. An analogous result is proven for injections of the complex projective spaces. In the case of continuity at a point this is generalized in several directions, the main one being that the complex numbers can be replaced by a finite-dimensional division algebra over an Archimedean ordered field. We also prove injective versions of the Fundamental Theorems of affine and projective geometry and give a counter-example to the surjective version of the latter. This extends work of A. G. Gorinov on a problem of V. I. Arnold.

1. Introduction

In his book of problems [1] V. I. Arnold asks if a homeomorphism, or more generally a bijection, $\mathbb{C}^n \to \mathbb{C}^n$ that sends affine subspaces to affine subspaces is necessarily either an affine mapping, or the complex conjugate of such a map, with a similar question being asked about homeomorphisms of the complex projective space $\mathbb{CP}^n$ and posing analogous questions about the quaternionic affine and projective spaces $\mathbb{H}^n$ and $\mathbb{HP}^n$. (Cf. [1] Problems 2000-8 (p. 134), 2002-9, 2002-10 (pp. 144–145) and the comments on these problems p. 614 and p. 627). For homeomorphisms A. G. Gorinov, [8], points out that in the case of $\mathbb{C}^n$ and $\mathbb{CP}^n$ the answer is affirmative and is a direct consequence of the fundamental theorem of projective geometry. He also shows the answer to a generalization of this question affirmative in the case of the quaternionic spaces.

In the cases of $\mathbb{C}^n$ and $\mathbb{CP}^n$ we extend these results in several directions. In the case of $\mathbb{C}^n$ the global continuity condition can be replaced either by the condition the map is continuous at least at one point or the condition the map is Lebesgue measurable.The condition that all affine subspaces are mapped to affine subspaces of the same dimension can be weakened to the condition that affine subspaces of some fixed dimension are mapped to affine subspaces (not necessarily of the same dimension). Also in this setting the map only needs to be an injection or surjection rather than a bijection. In the case where it is assumed that the map is only continuous at a single point, the complex numbers can be replaced by a finite-dimensional division algebra over an Archimedean ordered field. There are analogous results for $\mathbb{CP}^n$. Finally we prove as lemmas for our main results injective versions of the Fundamental Theorems of Affine and Projective Geometry (see Theorems 7 and 11) which may be of independent interest. We also show that

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the surjective analog of the Fundamental Theorem of Projective Geometry is not true, see Theorem 12 and Main Theorem 3.

2. Definitions and Statement of Main Results.

Unless stated otherwise, in this paper we let $\mathbb{D}$ be a division ring. All our division rings will be associative and with identity. We do not assume however that $\mathbb{D}$ has finite dimension over its center. Let $\mathbb{D}^n$ denote the (left) vector space of all n-tuples over $\mathbb{D}$ and $\mathbb{DP}^n$ projective space of dimension n over $\mathbb{D}$. (The points of $\mathbb{DP}^n$ are the one-dimensional left subspaces of $\mathbb{D}^{n+1}$.) A ***k-flat in*** $\mathbb{D}^n$ is a k-dimensional left affine subspace of $\mathbb{D}^n$ (that is, a translate of a k-dimensional left linear subspace of $\mathbb{D}^n$). Note that in the affine setting, the empty set is taken as a -1-flat. A ***k-flat in*** $\mathbb{DP}^n$ is a k-dimensional projective subspace of $\mathbb{DP}^n$.

Let σ be an automorphism of the division ring $\mathbb{D}$. A map $T\colon \mathbb{D}^n \to \mathbb{D}^n$ is ***σ-semilinear*** if and only if $T(x+y) = T(x) + T(y)$ and $T(cx) = \sigma(c)T(x)$ for all $x, y \in \mathbb{D}^n$ and $c \in \mathbb{D}$. A map $f\colon \mathbb{D}^n \to \mathbb{D}^n$ is ***σ-semiaffine*** if and only if it is of the form $f(x) = T(x) + b$ where $b \in \mathbb{D}^n$ and T is σ-semilinear. A map is ***semilinear*** (respectively ***semiaffine***) if and only if it is σ-semilinear (respectively σ-semiaffine) for some automorphism σ of $\mathbb{D}$. When σ is the identity map these are linear and affine maps.

These notions also apply to projective spaces. For a nonzero vector $v \in \mathbb{D}^{n+1}$ let $\langle v\rangle$ be the one-dimensional left subspace space spanned by v. Then $\langle v\rangle \in \mathbb{DP}^n$. If $A\colon \mathbb{D}^{n+1} \to \mathbb{D}^{n+1}$ is a nonsingular linear map then the map $T\colon \mathbb{DP}^n \to \mathbb{DP}^n$ defined by

$$T\langle v\rangle = \langle Av\rangle$$

will be called a ***linear map*** on $\mathbb{DP}^n$. If σ is an automorphism of $\mathbb{D}$ and A is σ-linear, the map T just defined is ***σ-linear***. We call T ***semilinear*** if it is σ-linear for some σ. In the case when $\mathbb{D} = \mathbb{C}$ and σ is complex conjugation, we refer to ***conjugate linear*** and ***conjugate affine*** maps.

Suppose $\mathbb{D}$ has been topologized in such a way that the difference and product operations $\mathbb{D}\times\mathbb{D} \to \mathbb{D}$, as well as the inversion $\mathbb{D}\smallsetminus\{0\} \to \mathbb{D}\smallsetminus\{0\}$ are continuous with respect to the natural topologies. Then $\mathbb{D}$ is called a ***topological division ring***, or a ***topological field*** if $\mathbb{D}$ is commutative. A straightforward check shows then that the product topology on a finite-dimensional $\mathbb{D}$-vector space $\mathbb{D}^n$ is invariant with respect to the group of affine transformations. In the sequel we equip all finite-dimensional affine spaces over $\mathbb{D}$ with the resulting topology.

A similar construction can be given for projective spaces. The subset $U = (\mathbb{D} \smallsetminus \{0\}) \times \mathbb{D}^{n-1} \subset \mathbb{D}^n$ is open and the map

$$(x_1, x_2, \dots, x_n) \mapsto (x_1^{-1}, x_2 \dots, x_n) \tag{1}$$

is a homeomorphism $U \to U$. The projective space space $\mathbb{DP}^n$ can be obtained by gluing copies of the affine space $\mathbb{D}^n$ along maps which are compositions of affine isomorphisms and (1). This gives us a topology on $\mathbb{DP}^n$ which is invariant with respect to the group of projective automorphisms.

Let $\mathbb{F}$ be a topological field. If a topological division ring $\mathbb{D}$ is an $\mathbb{F}$-algebra and the $\mathbb{F}$-module operation $\mathbb{F}\times\mathbb{D} \to \mathbb{D}$ is continuous, then $\mathbb{D}$ is a ***topological division*** ***$\mathbb{F}$-algebra***. In the sequel we topologize any finite-dimensional division algebra $\mathbb{D}'$ over $\mathbb{F}$ by embedding it in the endomorphism ring of the underlying $\mathbb{F}$-vector space. We note that with this topology $\mathbb{D}'$ is a topological division $\mathbb{F}$-algebra.

If $\mathbb{D} = \mathbb{F} = (\mathbb{F}, +, \cdot, <)$ is an ordered field, $\mathbb{F}$ has the ***order topology***, which is the topology that has the open intervals (a, b) as a base. In any ordered field there is the usual ***absolute value***, $|a| = \max\{a, -a\}$, and it satisfies the standard properties such as $|ab| = |a||b|$ and the triangle inequality. So a finite-dimensional division algebra $\mathbb{K}$ over $\mathbb{F}$ also has a natural topology, as do finite-dimensional affine and projective spaces over $\mathbb{K}$. If $\mathbb{K} = \mathbb{F}[\sqrt{-1}]$, one has a natural notion of complex conjugation, $a + b\sqrt{-1} \mapsto a - b\sqrt{-1}$, and therefore for the projective and affine spaces over $\mathbb{K}$ the notions of conjugate linear and conjugate affine make sense. Finally an ordered field $\mathbb{F}$ is ***Archimedean*** if for all $a \in \mathbb{F}$ there is a natural number n such that $|a| < n$. This is equivalent to the rational numbers being dense in $\mathbb{F}$ with respect to the order topology on $\mathbb{F}$.

For the rest of the paper $\mathbb{F}$ is an Archimedean ordered field and $\mathbb{K}$ is a topological division algebra over $\mathbb{F}$. We give $\mathbb{K}$ and all other finite-dimensional affine and projective spaces over $\mathbb{K}$ the topologies described above. Call an automorphism $\sigma\colon \mathbb{K} \to \mathbb{K}$ that fixes $\mathbb{F}$ pointwise an $\mathbb{F}$-***automorphism***. In the following when we say that T ***maps*** k-***flats into*** j-***flats*** we mean that for any k-flat P, there is a j-flat P' with $T[P] \subseteq P'$, but $T[P]$ might be a proper subset of P'. We say that T ***maps*** k-***flats to*** j-***flats*** if and only if for any k-flat P there is a j-flat P' such that $T[P] = P'$.

Main Theorem 1. *Let V be a finite-dimensional vector space over $\mathbb{K}$ with dimension at least 2. Let $T\colon V \to V$ be a map. If*

(i) *T is surjective,*
(ii) *T is continuous at some point of V or $\mathbb{F} = \mathbb{R}$, $\dim_{\mathbb{F}} \mathbb{K} < \infty$ and T is Lebesgue measurable, and*
(iii) *There is some k with $1 \leq k < \dim V$, such that T maps each k-flat into a k-flat,*

then T is a bijection and is σ-semiaffine for some $\mathbb{F}$-automorphism σ of $\mathbb{K}$. If $\mathbb{K} = \mathbb{F}[\sqrt{-1}]$, then T is either affine or conjugate affine.

Main Theorem 2 (Affine Version)**.** *Let V be a finite-dimensional vector space over $\mathbb{K}$ with dimension at least 2. Let $T\colon V \to V$ be a map. If*

(i) *T is injective,*
(ii) *T is continuous at some point of V or $\mathbb{F} = \mathbb{R}$, $\dim_{\mathbb{F}} \mathbb{K} < \infty$ and T is Lebesgue measurable, and*
(iii) *There is some k with $1 \leq k < \dim V$, such that T maps each k-flat to a flat,*

then T is a bijection and is σ-semiaffine for some $\mathbb{F}$-automorphism σ of $\mathbb{K}$. If $\mathbb{K} = \mathbb{F}[\sqrt{-1}]$, then T is either affine or conjugate affine.

Main Theorem 2 (Projective Version)**.** *Let V be a finite-dimensional projective space over $\mathbb{K}$ with projective dimension at least 2. Let $T\colon V \to V$ be a map. If*

(i) *T is injective,*
(ii) *T is continuous at some point of V or $\mathbb{F} = \mathbb{R}$, $\dim_{\mathbb{F}} \mathbb{K} < \infty$ and T is Lebesgue measurable, and*
(iii) *There is some k with $1 \leq k < \dim V$, such that T maps each k-flat to a flat,*

then T is a bijection and is σ-linear for some $\mathbb{F}$-automorphism σ of $\mathbb{K}$. If $\mathbb{K} = \mathbb{F}[\sqrt{-1}]$, then T is either linear or conjugate linear.

Main Theorem 3 (Counterexample for Projective Spaces)**.** *For every $n \geq 2$ there is a map $T\colon \mathbb{CP}^n \to \mathbb{CP}^n$ such that*

(i) *T is surjective,*
(ii) *Under the map T each point of $\mathbb{CP}^n$ is the image of infinitely many points and therefore T is not injective,*
(iii) *For all $m \in \{1, 2, \dots, n-1\}$ and each m-flat, P, of $\mathbb{CP}^n$ the image $T[P]$ is an m-flat of $\mathbb{CP}^n$,*
(iv) *For all $m \in \{1, 2, \dots, n-1\}$ every m-flat of $\mathbb{CP}^n$ is the image under T of an m-flat of $\mathbb{CP}^n$.*

3. Preliminary results.

3.1. **Additive Functions.** Let $S\colon V \to W$ be a map between vector spaces over a field $\mathbb{F}$. Then S is ***additive*** if and only if it satisfies

$$S(x+y) = S(x) + S(y) \quad \text{for all } x, y \in V. \tag{2}$$

This equation is often called ***Cauchy's functional equation*** after Cauchy who proved that an additive continuous map from the reals to the reals is linear.

One well-known extension of Cauchy's result is

Theorem 1 (Fréchet–Banach–Sierpiński)**.** *Let V and W be finite-dimensional real vector spaces and $S\colon V \to W$ an additive map. If S is Lebesgue measurable, then it is linear.* □

It is not hard to see that if this is true with $W = \mathbb{R}$, then it is true in general. (Write $S(v) = \sum_{j=1}^{n} f_j(v) w_j$ where $w_1, \dots, w_n$ is a basis of W. Then each f_j will be additive and measurable and therefore linear.) In the case when V is the real numbers, the theorem was originally proven by Fréchet [7]. It was later proven independently by Banach [3] and Sierpiński [14]. The proof given by Banach easily generalizes to the present case. A proof in the general case can also be found in Járai's book [10].

Another extension of Cauchy's result in the case of $S\colon \mathbb{R} \to \mathbb{R}$ is that if S is additive and continuous at a single point, then it is linear, a result due to Darboux [6] in 1875. We wish to extend this to maps between topological vector spaces over an ordered field $\mathbb{F}$. Recall that an $\mathbb{F}$-vector space V is ***topological*** if it has been equipped with a topology such that the sum operation $V \times V \to V$ and the $\mathbb{F}$-module structure map $\mathbb{F} \times V \to V$ are continuous. As an example, one can take V to be a finite-dimensional $\mathbb{F}$-vector space $\mathbb{F}^k$ with the product topology. For a $u \in \mathbb{F}^k$ we set $|u|_{\mathbb{F}^k} = \max_{i=1,\dots,k} |x_i|$. The sets $\{u \in \mathbb{F}^k : |u_0 - u|_{\mathbb{F}^k} < \varepsilon\}, \varepsilon \in \mathbb{F}, \varepsilon > 0$ are then a local base at $u_0 \in \mathbb{F}^k$.

Theorem 2 (An Extended Darboux's Theorem for Ordered Fields)**.** *Let $\mathbb{F}$ be an ordered field.*

If $\mathbb{F}$ is Archimedian, then for any topological vector spaces V, W over $\mathbb{F}$ every additive map from V to W which is continuous at a some point is a linear transformation.

If $\mathbb{F}$ is non-Archimedian, then for any positive integers m, n there is an additive continuous map $\mathbb{F}^m \to \mathbb{F}^n$ that is not $\mathbb{F}$-linear. (Here both $\mathbb{F}^m$ and $\mathbb{F}^n$ are given the product topologies.)

Proof. Suppose $\mathbb{F}$ is Archimedian and $f : V \to W$ is an additive map of topological $\mathbb{F}$-vector spaces that is continuous at some point. Since the topologies on V and W are translation invariant, f is in fact continuous everywhere. Moreover, for every $x \in V$ there is a continuous $\mathbb{F}$-linear map $l : \mathbb{F} \to V$ such that $l(1) = x$. This implies that the first part of the theorem for V arbitrary would follow from the same statement for $V = \mathbb{F}$. So let us assume $V = \mathbb{F}$. Then the map $g : \mathbb{F} \to W$ given by $g(x) = xf(1)$ is continuous, $\mathbb{F}$-linear and coincides with f on the dense subset $\mathbb{Q} \subset \mathbb{F}$, so by continuity f and g coincide everywhere.

Now suppose that $\mathbb{F}$ is not Archimedean, and let m, n be positive integers. Then $\mathbb{F}$ must have some infinitesimal elements other than 0. Let $\mathbb{I}$ be the set of all the m-tuples of infinitesimal elements of $\mathbb{F}$. Observe that both $\mathbb{I}$ and $V = \mathbb{F}^m$ are vector spaces over $\mathbb{Q}$. Let B_0 be a basis for $\mathbb{I}$ over $\mathbb{Q}$. Let $\mathbf{e} = (1, 0, 0, \ldots, 0) \in V$. Since $\mathbf{e} \notin \mathbb{I}$, the set $B_0 \cup \{\mathbf{e}\}$ is linearly independent over $\mathbb{Q}$. Extend this set to a basis B of V over $\mathbb{Q}$. Now let $\mathbf{e}^* = (1, 0 \ldots, 0) \in W$. Let S be the linear transformation (over $\mathbb{Q}$) from V to W defined, for all $u \in V$, by $S(u) = r\mathbf{e}^*$ where r is the coefficient of $\mathbf{e}$ when u is written as linear combination over B. Then S is certainly additive. However, were S linear over $\mathbb{F}$ we would have $S(a\mathbf{e}) = aS(\mathbf{e}) = a\mathbf{e}^*$ for all $a \in \mathbb{F}$. But $S(a\mathbf{e}) = 0$ whenever a is an infinitesimal element of $\mathbb{F}$—so $S(a\mathbf{e}) \neq a\mathbf{e}^*$ when a is a nonzero infinitesimal. Therefore S is not linear over $\mathbb{F}$. It remains to show that S is continuous. As shown above, it is enough to prove it is continuous at the zero vector. To this end, let $\varepsilon > 0$. We must produce a $\delta > 0$ so that for all $u \in \mathbb{F}^m$, if $|u|_{\mathbb{F}^m} < \delta$, then $|S(u)|_{\mathbb{F}^n} < \varepsilon$. Take δ to be any infinitesimal with $\delta > 0$. Then $|u|_{\mathbb{F}^m} < \delta$ entails that u is an m-tuple of infinitesimals. But then $|S(u)|_{\mathbb{F}^n} = |0|_{\mathbb{F}^n} < \varepsilon$. $\square$

Remark 3. In addition to continuity conditions, our Main Theorems have hypotheses concerning the surjectivity or injectivity of the maps involved. The map we constructed in the proof of the theorem above has neither of these properties. Nevertheless, we are unable eliminate the Archimedean hypothesis from our Main Theorems through the use of these additional hypotheses. Indeed, over any non-Archimedean ordered field on any finite-dimensional vector space there will always be continuous, additive, bijective maps that are not linear operators. Let S be the map produced in the proof of Theorem 2. Let $V = W = \mathbb{F}^n$ and let $a, b \in \mathbb{Q}$. Define $S_{a,b} \colon V \to W$ via

$$S_{a,b}(u) = aS(u) + bu \text{ for all } u \in V.$$

Evidently, each $S_{a,b}$ is continuous and additive. Recalling that S is a linear map, when $\mathbb{F}^n$ is construed as a vector space over $\mathbb{Q}$, it is easy to see that $S_{c,d}$ is the inverse of $S_{a,b}$ where $c = -a/(b(a+b))$ and $d = 1/b$, provided $b(a+b) \neq 0$. So, for example, $S_{1,1}(u) = S(u) + u$ is invertible and its inverse is $S_{-1/2,1}(u) = -1/2S(u) + u$. On the other hand, $S = 1/a(S_{a,b} - bI)$. Since we know that S is not linear over $\mathbb{F}$, we see that $S_{a,b}$ cannot be linear over $\mathbb{F}$ either.

In this way, we see that over any non-Archimedean ordered field continuous additive bijective functions need not be linear.

3.2. **Extended forms of the Fundamental Theorem of Affine and Projective Geometry.** Let $\mathbb{D}$ be a division ring. We use as our model of $\mathbb{DP}^n$, that is the n-dimensional projective space over $\mathbb{D}$, the space of one-dimensional left subspaces of $\mathbb{D}^{n+1}$. If $v \in \mathbb{D}^{n+1}$ with $v \neq 0$ let $\langle v \rangle$ be the one-dimensional left subspace spanned by v. If $A \colon \mathbb{D}^{n+1} \to \mathbb{D}^{n+1}$ is semilinear and nonsingular, then it induces a projective map $\hat{A}\colon \mathbb{DP}^n \to \mathbb{DP}^n$ by

$$\hat{A}\langle v\rangle = \langle Av\rangle.$$

For each $n \geq 2$, let $\mathbb{L}(\mathbb{D}^n)$ (respectively $\mathbb{L}(\mathbb{DP}^n)$) be the lattice of all flats in $\mathbb{D}^n$ (respectively $\mathbb{DP}^n$). The following are special cases of the Fundamental Theorems of Affine and Projective Geometry.

Theorem 4 (Fundamental Theorem of Affine Geometry)**.** *For $n \geq 2$, a bijection $T\colon \mathbb{D}^n \to \mathbb{D}^n$ induces an automorphism of $\mathbb{L}(\mathbb{D}^n)$ if and only if T is semiaffine.* $\square$

Theorem 5 (Fundamental Theorem of Projective Geometry)**.** *For $n \geq 2$, If a bijection $T\colon \mathbb{DP}^n \to \mathbb{DP}^n$ induces an automorphism of $\mathbb{L}(\mathbb{DP}^n)$, then there is a semilinear map $A\colon \mathbb{D}^{n+1} \to \mathbb{D}^{n+1}$ such that T is the induced map $T = \hat{A}$.* $\square$

This version of the Fundamental Theorem of Projective Geometry follows from [2, Thm 2.26 p. 88] and the Fundamental Theorem for Affine Geometry can be derived from the projective version. For a direct proof of the affine version see [4, pp. 201–202]

A version of the Fundamental Theorem of Affine Geometry where the assumption of the map T being bijective is replaced by T being surjective was proven by Alexander Chubarev and Iosif Pinelis [5]:

Theorem 6 (Surjective Fundamental Theorem of Affine Geometry)**.** *Let $\mathbb{D}$ and $\mathbb{D}'$ be division rings such that $\mathbb{D}$ has more than two elements. Let $\mathcal{A}$ and $\mathcal{A}'$ be affine spaces of finite dimensions n and n' over $\mathbb{D}$ and $\mathbb{D}'$ respectively and let $n' \geq n \geq 2$. If T is a map from $\mathcal{A}$ to $\mathcal{A}'$ such that*

(i) *T is surjective and*
(ii) *There is some k with $1 \leq k < n$ such that T maps each k-flat into a k-flat,*

then T is bijective and semiaffine. $\square$

The third stipulation in the affine version of our Main Theorem 2 only insists that the image of every k-flat is a flat and replaces surjectivity with injectivity.

Theorem 7 (Injective Fundamental Theorem of Affine Geometry)**.** *Let $\mathbb{D}$ be a division ring with more than two elements. Let $\mathcal{A}$ be the affine space of finite dimension $n > 1$ over $\mathbb{D}$. If T is a map from $\mathcal{A}$ to $\mathcal{A}$ such that*

(i) *T is injective and*
(ii) *There is some k with $1 \leq k < n$ such that T maps each k-flat to a flat,*

then T is bijective and semiaffine.

Remark 8. If $\mathbb{D} = \mathbb{Z}/2$ and $\mathcal{A}, T, n, k$ are as in Theorem 7, then by [5, Proposition 1] the conclusion of the theorem remains true provided $n = 2$ or $k \geq 2$.

Proposition 9. *Let $\mathbb{D}$ be a division ring, and $n \geq 2$ be a natural number. Let V be either the affine space or the projective space of dimension n over $\mathbb{D}$. If V is affine, we assume moreover that $\mathbb{D}$ contains more than two elements. If T is a map from V to V such that*

(i) *T is injective, and*
(ii) *there is some k with $1 \leq k < n$ such that T maps each k-flat to a flat,*

then T is bijective and the map $Q \mapsto T[Q]$ is an automorphism of the lattice $\mathbb{L}(V)$.

Lemma 10. *Under the hypotheses of Proposition 9, T maps every flat to a flat. Moreover T is bijective and for every $k \leq n$, T maps every k-flat to a k-flat.*

Proof. We leave the "moreover" portion of the lemma to the end of this proof. We consider three cases.

Case: $\dim Q = k$.
It is one of our hypotheses that $T[Q]$ is a flat.

Case: $\dim Q < k$.
There is a finite set $\{P_0, P_2, \ldots, P_{m-1}\}$ of k-flats such that

$$Q = \bigcap_{j<m} P_j.$$

As T is injective, $T[Q] = \bigcap_{j<m} T[P_j]$. Each $T[P_j]$ is a flat and therefore $T[Q]$ is the intersection of flats and thus is itself a flat.

Case: $\dim Q > k$.
We use the fact that under our assumptions on $\mathbb{D}$ and V a subset of V is a flat if and only if it contains the line through any two of its points. Let $y_1, y_2 \in T[Q]$ be distinct. Then there are distinct points $x_1, x_2 \in Q$ with $y_1 = T(x_1)$ and $y_2 = T(x_2)$. As $\dim Q > k \geq 1$ there is a flat $P \subset Q$ with $\dim P = k$ and $x_1, x_2 \in P$. The image $T[P]$ is a flat and contains the points $y_1 = T(x_1)$ and $y_2 = T(x_2)$ and thus contains the line through y_1 and y_2. As $T[P] \subset T[Q]$ this shows that $T[Q]$ contains the line through y_1 and y_2 and as these are arbitrary points of $T[Q]$, we have that $T[Q]$ is a flat.

Now let us consider the "moreover" portion of the Lemma. Let $\ell = \dim Q$. There is a strictly increasing chain $Q_0 \subsetneq Q_1 \subsetneq \cdots \subsetneq Q_n = \mathbb{D}^n$ of flats such that $\dim Q_j = j$ and $Q_\ell = Q$. Here the flat Q_0 can be any point of Q. Then

$$T[Q_0] \subsetneq T[Q_1] \subsetneq \cdots \subsetneq T[Q_n]$$

is a strictly increasing chain of flats of $\mathbb{D}^n$, since T is injective. As $T[Q_{j+1}]$ strictly contains $T[Q_j]$ the inequality $\dim T[Q_{j+1}] \geq 1+\dim T[Q_j]$ holds. Thus $\dim T[Q_j] \geq j$ for all j. So that $\dim T[Q_n] \geq n$. But $\mathbb{D}^n$ has only one flat with dimension at least n and that is $\mathbb{D}^n$ itself. So $\dim T[Q_n] = n$. This is only possible if $\dim T[Q_j] = j$ for all $j \leq n$. In particular $\dim T[Q] = \dim T[Q_\ell] = \ell = \dim Q$, as required. Finally $T[\mathbb{D}^n]$ is flat of dimension n and therefore $T[\mathbb{D}^n] = \mathbb{D}^n$ which shows that T is surjective and therefore bijective. □

Proof of Proposition 9. By the lemma above and the injectivity of T the map $P \mapsto T[P]$ is an injective map from $\mathbb{L}(V)$ to itself that preserves the lattice operations. All that remains is to show that this map is surjective. As $T[T^{-1}[Q]] = Q$ it is enough to show that $T^{-1}[Q]$ is a flat whenever Q is a flat. If $\dim Q = 0$ this is clear. So assume $\dim Q \geq 1$. Let $x_1, x_2 \in T^{-1}[Q]$ be distinct and let L be the line through x_1 and x_2. Then by the lemma $T[L]$ is a line and, as it contains the points $T(x_1)$ and $T(x_2)$ of the flat Q, the line $T[L]$ is contained in Q. Thus $L = T^{-1}[T[L]] \subseteq T^{-1}[Q]$. Therefore $T^{-1}[Q]$ contains the line through any two of its points and hence it is a flat. □

The Injective Fundamental Theorem of Affine Geometry follows from Theorem 4 and Proposition 9. □

3.3. **Extended form of the Fundamental Theorem Projective Geometry.**

Theorem 11 (Injective Fundamental Theorem of Projective Geometry)**.** *Let $\mathbb{D}$ be a division ring. Let V be the projective space of finite dimension $n > 1$ over $\mathbb{D}$. If T is a map from V to V such that*

(i) *T is injective and*
(ii) *There is some k with $1 \leq k < n$ such that T maps each k-flat to a flat,*

then T is bijective and semilinear.

Proof. We apply Proposition 9 and Theorem 5. □

We initially believed there was a projective analog of the Surjective Fundamental Theorem of Affine Geometry (Theorem 6). Rather surprisingly, it fails even for the complex projective plane. If $\mathbb{K}$ is a field let $\mathbb{K}\langle\langle t\rangle\rangle$ be the field of formal Puiseux series over $\mathbb{K}$ (see Section 5 for a precise description.)

Theorem 12 (Counterexample to Surjective Fundamental Theorem of Projective Geometry)**.** *Let $\mathbf{K}$ be an algebraically closed field of characteristic zero such that $\mathbf{K}\langle\langle t\rangle\rangle$ is isomorphic to $\mathbf{K}$ (e.g. the complex numbers). Then for any integer $n \geq 2$ there is a map $T\colon \mathbf{K}\mathbb{P}^n \to \mathbf{K}\mathbb{P}^n$ such that*

(i) *T is surjective,*
(ii) *each point $y \in \mathbf{K}\mathbb{P}^n$ is the image of infinitely points under T and so T is not injective,*
(iii) *for all $m \in \{1, 2, \ldots, n-1\}$ and every m-flat P of $\mathbf{K}\mathbb{P}^n$ the image $T[P]$ is a m-flat in $\mathbf{K}\mathbb{P}^n$, and*
(iv) *for all $m \in \{1, 2, \ldots, n-1\}$ every m-flat of $\mathbf{K}\mathbb{P}^n$ is the image of some m-flat under T.*

The key geometric ideas involved in the proof of this result (cf. Section 5) are based on ideas from Examples 1 and 2 (pages 377–378) from the paper by J. F. Rigby [13], however the algebraic details are substantially more complicated.

4. Proof of the main theorems.

Let $n \geq 2$ be an integer. Recall that $\mathbb{F}$ is an Archimedian ordered field, and $\mathbb{K}$ is a topological division algebra over $\mathbb{F}$, see Section 2, where we also describe the topologies on $\mathbb{K}^n$ and $\mathbb{K}\mathbb{P}^n$.

4.1. **Proof of the affine results.**

Consider a map T that fulfills the hypotheses of either of the affine versions of our Main Theorems. Using either the Surjective or Injective Fundamental Theorems of Affine Geometry, we see that T must be semiaffine. Thus there is an element $b \in \mathbb{K}^n$, an automorphism σ of $\mathbb{K}$ and a map $S\colon \mathbb{K}^n \to \mathbb{K}^n$ such that

$$\begin{aligned} S(x+y) &= S(x) + S(y) && \text{for all } x, y \in \mathbb{K}^n \\ S(cx) &= \sigma(c)S(x) && \text{for all } x \in \mathbb{K}^n \text{ and } c \in \mathbb{K} \\ T(x) &= S(x) + b && \text{for all } x \in \mathbb{K}^n \end{aligned}$$

Since T fulfills hypothesis (2) of our Main Theorems, so must S. Observe that $\mathbb{K}^n$ with the product topology is a topological vector space over $\mathbb{F}$. We conclude using

Theorems 2 and 1 that S is a linear operator on $\mathbb{K}^n$ considered as a vector space over $\mathbb{F}$. So for every element $r \in \mathbb{F}$ and for every $x \in \mathbb{K}^n$ we have

$$rS(x) = S(rx) = \sigma(r)S(x).$$

Because S does not map everything to the zero-vector, we find

$$r = \sigma(r) \quad \text{for all} \quad r \in \mathbb{F}.$$

Therefore σ is an $\mathbb{F}$-automorphism as required.

Finally assume $\mathbb{K} = \mathbb{F}[\sqrt{-1}]$. Observe that $x^2 + 1$ is irreducible over $\mathbb{F}$ and $\mathbb{K}$ is a degree two extension of $\mathbb{F}$ that contains both the roots of this polynomial. Thus $\mathbb{K}$ is the splitting field of $x^2 + 1$. It follows that the Galois group of $\mathbb{K}$ over $\mathbb{F}$ is just the two element group. So there are only two automorphisms of $\mathbb{K}$ that fix each member of $\mathbb{F}$: the identity map and conjugation. In the first alternative, T will be an affine map, while in the second alternative T will be conjugate-affine. This completes the proofs.

4.2. **Proof of the projective result.** Let $T\colon V \to V$ be a map that satisfies the hypotheses of projective version of Main Theorem 2. Then by the Injective Fundamental Theorem of Projective Geometry, we see that T is semilinear. Let H be any hyperplane in V. Then $T[H]$ is also a hyperplane in V. As the group of linear automorphisms of V is transitive on the set of hyperplanes, there is a linear automorphism S of V such that $S[T[H]] = H$. But then $S \circ T$ maps $V \smallsetminus H$ onto itself and $V \smallsetminus H$ is an affine space. Therefore by the affine versions of our Main Theorems, the restriction $(S \circ T)\big|_{V \smallsetminus H}\colon (V \smallsetminus H) \to (V \smallsetminus H)$ is σ-semiaffine for an $\mathbb{F}$-automorphism σ of $\mathbb{K}$. As S is linear this implies that $T\big|_{V \smallsetminus H} = S^{-1} \circ (S \circ T)\big|_{V \smallsetminus H}$ is σ-linear. From this it is not hard to check that T is σ-linear as required. If $K = \mathbb{F}[\sqrt{-1}]$, then S is either affine or conjugate affine, which implies that T is either linear or conjugate linear. □

5. Examples

5.1. **Algebraic preliminaries on Puiseux series.** Let $\mathbf{K}$ be a field of characteristic zero. For any variable x we denote by $\mathbf{K}((x))$ the field of formal Laurent series in x. Thus if $f(x) \in \mathbf{K}((x))$ is not the zero element, there is an unique integer k such that $f(x)$ is of the form

$$f(x) = \sum_{j=k}^{\infty} f_j x^j$$

where $f_j \in \mathbf{K}$ and $f_k \neq 0$. The integer k is the ***order*** $f(x)$ and is denoted by $\operatorname{ord}(f(x))$. For completeness we define $\operatorname{ord}(0) = +\infty$. In analogy with complex analysis the order of $f(x)$ can be thought of as order of the zero of $f(x)$ at the origin, with the usual convention that when $\operatorname{ord}(f(x))$ is negative then the origin is a pole. If $\operatorname{ord}(f(x)) \geq 0$, then we can evaluate $f(x)$ at $x = 0$ giving $f(0) = f_0$, the coefficient of 1 in the series $f(x) = \sum_j f_j x^j$. For a nonzero $f(x) \in \mathbf{K}((x))$ the coefficient of $x^{\operatorname{ord}(f(x))}$ is the ***lead coefficient*** of $f(x)$ and we will denote it by $\operatorname{lead}(f(x))$. Set $\operatorname{lead}(0) = 0$. With these definitions it is not hard to check for $f(x), g(x) \in \mathbf{K}((x))$ that

$$\operatorname{ord}(f(x)g(x)) = \operatorname{ord}(f(x)) + \operatorname{ord}(g(x)), \tag{3}$$

$$\operatorname{lead}(f(x)g(x)) = \operatorname{lead}(f(x)) \operatorname{lead}(g(x)). \tag{4}$$

If $\mathbf{K}[[x]]$ is the ring of formal power series over $\mathbf{K}$, then $\mathbf{K}((x))$ is the ring of fractions of $\mathbf{K}[[x]]$. Note also that $\mathbf{K}[[x]]$ is a principal ideal domain (the ideals are all of the form (x^m) for some nonnegative integer m).

Let n be a positive integer. A variant on the above is $\mathbf{K}((x^{1/n}))$, the field of formal Laurent series in $x^{1/n}$. In this case the order of a nonzero $f(x) \in \mathbf{K}((x^{1/n}))$ is still defined as the smallest exponent of a nonzero term in the sum $f(x) = \sum_{r\in\frac{1}{n}\mathbb{Z}} f_r x^r$ where $\frac{1}{n}\mathbb{Z} = \{k/n : k \in \mathbb{Z}\}$. Thus in this case $\operatorname{ord}(f(x))$ is a rational number of the form k/n where k is an integer. Likewise the lead coefficient, $\operatorname{lead}(f(x))$, is still defined and if we still use the convention that $\operatorname{ord}(0) = +\infty$ and $\operatorname{lead}(0) = 0$, the formulas (3) and (4) still hold. Also $\mathbf{K}((x^{1/n}))$ is the ring of fractions of $\mathbf{K}[[x^{1/n}]]$ and $\mathbf{K}[[x^{1/n}]]$ is a principal ideal domain.

Finally the ***field of formal Puiseux series*** over $\mathbf{K}$ is the union

$$\mathbf{K}\langle\langle x\rangle\rangle = \bigcup_{n=1}^{\infty} \mathbf{K}((x^{1/n})).$$

For $f(x) \in \mathbf{K}\langle\langle x\rangle\rangle$ the $\operatorname{ord}(f(x))$ and $\operatorname{lead}(f(x))$ are defined and satisfy (3) and (4). If $\operatorname{ord}(f(x)) \geq 0$ then the evaluation, $f(0)$, is defined in the natural way. Evaluation and the lead coefficient are related as follows. If $\operatorname{ord}(f(x)) = k$ then $f(x)$ is of the form $f(x) = x^k \widetilde{f}(x)$ where $\operatorname{ord}(\widetilde{f}(x)) = 0$. Then

$$\operatorname{lead}(f(x)) = \widetilde{f}(0).$$

We will need the following result on the algebraic closure of the field $\mathbf{K}((x))$.

Theorem 13 (The Newton-Puiseux Theorem)**.** *If $\mathbf{K}$ is an algebraically closed field of characteristic zero, then $\mathbf{K}\langle\langle x\rangle\rangle$ is an algebraic closure of $\mathbf{K}((x))$.* $\square$

See e.g. [16, pp. 98–102] for a proof. Newton's and Puiseux's original versions can be found in [11], [12].

5.2. **Construction of the examples.** Let $\mathbf{F}$ be a field. Then in this section we use the notation $\mathbb{P}^n(\mathbf{F})$ for the projective space $\mathbf{F}\mathbb{P}^n$ realized as set of points with homogeneous coordinates $[a_0 : a_1 : \cdots : a_n]$. If $\mathbf{a} = (a_0, a_1, \ldots, a_n) \in \mathbf{F}^{n+1} \smallsetminus \{0\}$ let $[\mathbf{a}]$ be the point in $\mathbb{P}^n(\mathbf{F})$ with homogeneous coordinates $[a_0 : a_1 : \cdots : a_n]$. Then for $\mathbf{a}, \mathbf{b} \in \mathbf{F}^{n+1} \smallsetminus \{0\}$ we have $[\mathbf{a}] = [\mathbf{b}]$ if and only if $\mathbf{a} = \lambda\mathbf{b}$ for some nonzero $\lambda \in \mathbf{F}$.

Let $\mathbf{a}(x) = (a_0(x), a_1(x), \ldots, a_n(x))$ be an $(n+1)$-tuple of elements from $\mathbf{K}\langle\langle x\rangle\rangle$. We extend the definition of ord to such tuples by

$$\operatorname{ord}(\mathbf{a}(x)) = \min_{0\leq j\leq n} \operatorname{ord}(a_j(x)).$$

As in the case of elements of $\mathbf{K}\langle\langle x\rangle\rangle$ if $\operatorname{ord}(\mathbf{a}(x)) \geq 0$, we can evaluate $\mathbf{a}(x)$ at zero by

$$\mathbf{a}(x) = (a_0(0), a_1(0), \ldots, a_n(0)).$$

If $\mathbf{a}(x) \in \mathbf{K}\langle\langle x\rangle\rangle^{n+1} \smallsetminus \{0\}$, write

$$\mathbf{a}(x) = x^{\operatorname{ord}(\mathbf{a}(x))}\widetilde{\mathbf{a}}(x)$$

where $\operatorname{ord}(\widetilde{\mathbf{a}}(x)) = 0$. As $\widetilde{\mathbf{a}}(x)$ has order zero, the evaluation $\widetilde{\mathbf{a}}(0) \in \mathbf{K}^{n+1}$ satisfies $\widetilde{\mathbf{a}}(0) \neq 0$. Define the ***lead coefficient*** of $\mathbf{a}(x)$ by

$$\operatorname{lead}(\mathbf{a}(x)) = \widetilde{\mathbf{a}}(0).$$

Set $\operatorname{lead}(0) = 0$. The proof of the following is left to the reader.

Lemma 14. *If $\lambda(x) \in \mathbf{K}\langle\langle x\rangle\rangle$ and $\mathbf{a}(x) = (a_0(x), a_1(x), \dots, a_n(x))$ is an $(n+1)$-tuple of elements from $\mathbf{K}\langle\langle x\rangle\rangle$, then the equations*

$$\operatorname{ord}(\lambda(x)\mathbf{a}(x)) = \operatorname{ord}(\lambda(x)) + \operatorname{ord}(\mathbf{a}(x))$$
$$\operatorname{lead}(\lambda(x)\mathbf{a}(x)) = \operatorname{lead}(\lambda(x))\operatorname{lead}(\mathbf{a}(x))$$

hold. □

Definition 15. If $\mathbf{K}$ is a field of characteristic zero and n a positive integer, the ***lead coefficient map*** is the function $L\colon \mathbb{P}^n(\mathbf{K}\langle\langle x\rangle\rangle) \to \mathbb{P}^n(\mathbf{K})$ given by

$$L([\mathbf{a}(x)]) = [\operatorname{lead}(\mathbf{a}(x))]$$

(This is well defined by Lemma 14). □

Proposition 16. *Let $\mathbf{K}$ be a field of characteristic zero, and let n be an integer ≥ 2. Then the lead coefficient map $L\colon \mathbb{P}^n(\mathbf{K}\langle\langle x\rangle\rangle) \to \mathbb{P}^n(\mathbf{K})$ is surjective. Any $[\mathbf{b}] \in \mathbb{P}^n(\mathbf{K})$ is the image under L of infinitely many elements of $\mathbb{P}^n(\mathbf{K}\langle\langle x\rangle\rangle)$ and thus L is not injective. For for every m-flat, P, of $\mathbb{P}^n(\mathbf{K}\langle\langle x\rangle\rangle)$, the image $L[P]$ is an m-flat in $\mathbb{P}^n(\mathbf{K})$. Moreover every m-flat in $\mathbb{P}^n(\mathbf{K})$ is the image under L of some m-flat of $\mathbb{P}^n(\mathbf{K}\langle\langle x\rangle\rangle)$.*

Proof. Let $[\mathbf{a}] = [(a_0, a_1, \dots, a_n)] \in \mathbb{P}^n(\mathbf{K})$. Choose any $b_0(x), b_1(x), \dots, b_n(x) \in \mathbf{K}\langle\langle x\rangle\rangle$ such that $\operatorname{ord}(b_j(x)) > 0$ for $j \in \{0, 1, \dots, n\}$. Then

$$L([(a_0 + b_0(x), a_1 + b_1(x), \dots, a_n + b_n(x))]) = [\mathbf{a}].$$

Thus L is surjective. There are infinitely many choices for $b_0(x), b_1(x), \dots, b_n(x)$, thus any point of $\mathbb{P}^n(\mathbf{K})$ is the image of infinitely many points of $\mathbb{P}^n(\mathbf{K}\langle\langle x\rangle\rangle)$.

Every m-dimensional projective subspace of $\mathbb{P}(\mathbf{K}\langle\langle x\rangle\rangle)$ is of the form $\mathbb{P}(V)$ for an $(m+1)$-dimensional vector subspace V of $\mathbf{K}\langle\langle x\rangle\rangle^{n+1}$.

Claim 1. The subspace V has a basis $\mathbf{v}_0(x), \mathbf{v}_1(x), \dots, \mathbf{v}_m(x)$ such that each element of the basis has $\operatorname{ord}(\mathbf{v}_j(x)) = 0$ and the vectors $\mathbf{v}_0(0), \mathbf{v}_1(0), \dots, \mathbf{v}_m(0)$ are linearly independent in $\mathbf{K}^{n+1}$.

To see this start with any basis $\mathbf{a}_0(x), \mathbf{a}_1(x), \dots, \mathbf{a}_m(x)$ of V. If need be, we can replace $\mathbf{a}_0(x)$ by $x^{-\operatorname{ord}(\mathbf{a}_0(x))}\mathbf{a}_0(x)$ and assume that $\mathbf{a}_0(x)$ has $\operatorname{ord}(\mathbf{a}_0(x)) = 0$. Form the matrix $A(x)$ that has $\mathbf{a}_0(x), \mathbf{a}_1(x), \dots, \mathbf{a}_m(x)$ as rows:

$$A(x) = \begin{bmatrix} \mathbf{a}_0(x) \\ \mathbf{a}_1(x) \\ \vdots \\ \mathbf{a}_m(x) \end{bmatrix} = \begin{bmatrix} a_{00}(x) & a_{01}(x) & \cdots & a_{0n}(x) \\ a_{10}(x) & a_{11}(x) & \cdots & a_{1n}(x) \\ \vdots & \vdots & \ddots & \vdots \\ a_{m0}(x) & a_{m0}(x) & \cdots & a_{mn}(x) \end{bmatrix}.$$

By doing a permutation, σ_0, of the columns we can assume that $\operatorname{ord}(a_{00}(x)) = 0$. Multiplying the first row by $a_{00}(x)^{-1}$ we can assume that $a_{00}(x) = 1$. Then by doing elementary row operations (replacing $\mathbf{a}_j(x)$ by $\mathbf{a}_j(x) - a_{j0}(x)\mathbf{a}_0(x)$) we get a matrix where all the elements of the first column other than the first element are zero:

$$A_1(x) = \begin{bmatrix} 1 & b_{01}(x) & \cdots & b_{0n}(x) \\ 0 & b_{11}(x) & \cdots & b_{1n}(x) \\ \vdots & \vdots & \ddots & \vdots \\ 0 & b_{m0}(x) & \cdots & b_{mn}(x) \end{bmatrix}$$

and all the elements of the first row have order at least zero. Because of the permutation σ_0 the rows of this matrix need not be in the subspace V, but applying the inverse σ_0^{-1} to the columns of $A_1(x)$ leads to a matrix that differs from the original matrix $A(x)$ by the application of elementary row operations and therefore its rows will be a basis of V.

Continuing in this manner (using column permutations $\sigma_1, \cdots, \sigma_{m-1}$ and elementary row operations) $A(x)$ can be reduced to the form

$$A_m(x) = \begin{bmatrix} 1 & * & * & * & * & \cdots & * \\ 0 & 1 & * & * & * & \cdots & * \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 & * & \cdots & * \end{bmatrix}$$

where all the elements represented by $*$ have order at least zero. By applying the permutation $(\sigma_{m-1} \cdots \sigma_1 \sigma_0)^{-1}$ to the columns of this matrix we get a matrix $A'(x)$ that is derived from the original matrix by elementary row operations and therefore the rows of $A'(x)$ are a basis of V. The rows of $A_m(0)$ are clearly linearly independent and $A'(0)$ differs from $A_m(0)$ a permutation of the columns and therefore the rows of $A'(0)$ are linearly independent. Thus if $\mathbf{v}_0(x), \mathbf{v}_1(x), \dots, \mathbf{v}_m(x)$ are the rows of $A'(x)$, then they are a basis of V such that $\mathbf{v}_0(0), \mathbf{v}_1(0), \dots, \mathbf{v}_m(0)$ are linearly independent in $\mathbf{K}^{n+1}$, which verifies the claim.

Dually, the vector subspace V of $\mathbf{K}^{n+1}$ could be given as the solution set of $n-m$ linearly independent linear equations

$$V = \Big\{ \mathbf{a}(x) : \ell_i(\mathbf{a}(x)) = 0 \text{ for } i \in \{1, 2, \dots, n-m\} \Big\}$$

where ℓ_i is of the form

$$\ell_i(\mathbf{a}(x)) = c_{i0}(x)a_0(x) + c_{i1}(x)a_1(x) + \cdots + c_{in}(x)a_n(x). \tag{5}$$

Claim 2. It is possible to choose the linear functions such that all the coefficients have order at least zero and such that the matrix $[c_{ij}(0)]$ has rank $n-m$ over $\mathbf{K}$ and therefore the linear functionals on $\mathbf{K}^{n-1}$ defined by $\ell_i'(\mathbf{a}) = c_{i0}(0)a_0 + c_{i1}(0)a_1 + \cdots + c_{in}(0)a_n$ are linearly independent over $\mathbf{K}$.

The proof of this claim is almost identical to the proof of Claim 1. Form the matrix $C(x) = [c_{ij}(x)]$ and perform the same elementary row operations as in the proof of the first claim to get a matrix $C'(x)$ such that the rows of $C'(x)$ have the same span as those of $C(x)$ and such that $C'(0)$ has linearly independent rows in $\mathbf{K}^{n+1}$. Then using the i-th row of $C'(x)$ as the coefficients of $\ell_i(x)$ completes the argument.

Returning to the proof of Proposition 16, let $P = \mathbb{P}(V)$ be an m-flat in $\mathbb{P}(\mathbf{K}\langle\langle x\rangle\rangle^{n+1})$. Choose a basis $\mathbf{v}_0(x), \mathbf{v}_1(x), \dots, \mathbf{v}_m(x)$ as in Claim 1. Let V' be the subspace of $\mathbf{K}^{n+1}$ with basis $\mathbf{v}_0(0), \mathbf{v}_1(0), \dots, \mathbf{v}_m(0)$. Then a chase through the definition of L shows

$$\mathbb{P}(V') \subseteq L[P] = L[\mathbb{P}(V)].$$

For the reverse inclusion we let $\ell_1, \ell_2, \dots, \ell_{n-m}$ be the linear functionals on $\mathbf{K}\langle\langle x\rangle\rangle^{n+1}$ given by Claim 2 and let $[\mathbf{a}(x)] \in \mathbb{P}(V)$. Without loss of generality assume $\operatorname{ord} \mathbf{a}(x) = 0$. Then $\mathbf{a}(0)$ is defined and $\mathbf{a}(0) \neq \mathbf{0}$. Then $\mathbf{a}(x) \in V$ and thus $\ell_i(\mathbf{a}(x)) = 0$. Let ℓ_i' be the linear functional on $\mathbf{K}^{n+1}$ obtained by evaluating the coefficients of ℓ_i at $x = 0$ as in Claim 2. Then Claim 2 yields that $\ell_1', \ell_2', \dots, \ell_{n-m}'$ are linearly independent linear functionals on $\mathbf{K}^{n+1}$. Evaluating $\ell_i(\mathbf{a}(x)) = 0$ at

$x = 0$ shows $\ell_i'(\mathbf{a}(0)) = 0$. Therefore $\mathbf{a}(0) \in \bigcap_{i=1}^{n-m} \ker(\ell_i')$. Set $V'' = \bigcap_{i=1}^{n-m} \ker(\ell_i')$. As $[\mathbf{a}(x)]$ was any element of $\mathbb{P}(V)$ this yields

$$\mathbb{P}(V') \subseteq L[P] = L[\mathbb{P}(V)] \subseteq \mathbb{P}(V'').$$

Comparing dimensions shows $V' = V''$ and therefore $L[P] = \mathbb{P}(V')$ which shows $L[P]$ is an m-flat of $\mathbb{P}(\mathbf{K}^{n+1})$.

Finally let $P' = \mathbb{P}(V')$ be an m-flat in $\mathbb{P}(\mathbf{K}^{n+1})$ and let $\mathbf{v}_0, \mathbf{v}_1, \ldots, \mathbf{v}_m$ be a basis of V'. We can view $\mathbf{v}_0, \mathbf{v}_1, \ldots, \mathbf{v}_m$ as elements of $\mathbf{K}\langle\langle x\rangle\rangle^{n+1}$ and let V be the span of these vectors in $\mathbf{K}\langle\langle x\rangle\rangle^{n+1}$. Then P' is the image under L of $P = \mathbb{P}(V)$. Thus every m-flat of $\mathbb{P}(\mathbf{K}^{n+1})$ is the image of an m-flat of $\mathbb{P}(\mathbf{K}\langle\langle x\rangle\rangle^{n+1})$. □

Proof of Theorem 12. Theorem 12 follows immediately from Proposition 16.

□

5.2.1. *Proof of Main Theorem 3.* To apply Theorem 12 we need to give conditions that insure that $\mathbf{K}$ and $\mathbf{K}\langle\langle x\rangle\rangle$ are isomorphic. If K is a field, we denote the transcendence degree of K over its prime subfield by $\operatorname{trdeg} K$. Note that if $\operatorname{trdeg} K$ is infinite, then it is equal the cardinality $|K|$ of K. A basic result in the theory of transcendental field extensions is the theorem of Steinitz that states that two algebraically closed fields K_1, K_2 are isomorphic if and only if $\operatorname{char} K_1 = \operatorname{char} K_2$ and $\operatorname{trdeg} K_1 = \operatorname{trdeg} K_2$ (cf. [15]).

Let $\mathbf{K}$ be an algebraically closed of characteristic zero that has infinite transcendence degree over the rationals. We note that the set $\mathbf{K}((x))$ of formal Laurent series has cardinality $|\mathbf{K}((x))| = |\mathbf{K}|^{\aleph_0}$. But $\mathbf{K}\langle\langle x\rangle\rangle = \bigcup_{n=1}^{\infty} \mathbf{K}((x^{1/n}))$, and thus $|\mathbf{K}\langle\langle x\rangle\rangle| = \aleph_0 \cdot |\mathbf{K}|^{\aleph_0} = |\mathbf{K}|^{\aleph_0}$. So if $|\mathbf{K}|^{\aleph_0} = |\mathbf{K}|$, then

$$\operatorname{trdeg}(\mathbf{K}\langle\langle x\rangle\rangle) = |\mathbf{K}\langle\langle x\rangle\rangle| = |\mathbf{K}|^{\aleph_0} = |\mathbf{K}| = \operatorname{trdeg}(\mathbf{K}),$$

which implies that $\mathbf{K}$ and $\mathbf{K}\langle\langle x\rangle\rangle$ are isomorphic.

As an example, suppose $|\mathbf{K}| = \lambda^\mu$ where λ and μ are cardinals and μ is infinite. Note that this includes the case $\mathbf{K} = \mathbb{C}$. Then we have

$$|\mathbf{K}|^{\aleph_0} = (\lambda^\mu)^{\aleph_0} = \lambda^{\mu\cdot\aleph_0} = \lambda^\mu = |\mathbf{K}|.$$

Summarizing:

Proposition 17. *Let $\mathbf{K}$ be an algebraically closed field of characteristic zero. Then the fields $\mathbf{K}$ and $\mathbf{K}\langle\langle x\rangle\rangle$ are isomorphic if and only if $|\mathbf{K}|^{\aleph_0} = |\mathbf{K}|$. The latter condition is satisfied for $\mathbf{K} = \mathbb{C}$ or, more generally, if $|\mathbf{K}| = \lambda^\mu$ for some cardinals λ, μ such that μ is infinite.* □

Main Theorem 3 now follows from Proposition 17 and Theorem 12.

Remark 18. There exist algebraically closed fields of characteristic zero such that the cardinality of the field of formal Puiseux series is strictly larger than the cardinality of the field itself. One such example is $\overline{\mathbb{Q}}$, the algebraic closure of $\mathbb{Q}$. To construct other examples recall that the **cofinality** of a cardinal κ is the least cardinality of a cofinal subset of the set $[0, \kappa)$ of cardinals. Let α be a countable limit ordinal. Take β to be an arbitrary ordinal and set $\kappa = \aleph_{\beta+\alpha}$. The cofinality of κ is $\aleph_0$. It follows from König's inequality that $\kappa^{\aleph_0} > \kappa$, see e.g. [9, Theorem 1.6.9]. So if we take $\mathbf{K}$ to be an algebraically closed field of characteristic zero with $\operatorname{trdeg} = \kappa$, we get $|\mathbf{K}\langle\langle x\rangle\rangle| > |\mathbf{K}|$.

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