Abstract

Let G be a Lie group and K a compact subgroup of G. Then the homogeneous space G/K has an invariant Riemannian metric and an invariant volume form Ω_G. Let M and N be compact submanifolds of G/K, and $I(M \cap gN)$ an "integral invariant" of the intersection $M \cap gN$. Then the integral

$$\int_G I(M \cap gN)\, \Omega_G(g) \tag{1}$$

is evaluated for a large class of integral invariants I. To give an informal definition of the integral invariants I considered, let $X \subset G/K$ be a submanifold, h^X the vector valued second fundamental form of X in G/K. Let $\mathcal{P}$ be an "invariant polynomial" in the components of the second fundamental form of h^X. Then the integral invariants considered are of the form

$$I^{\mathcal{P}}(X) = \int_X \mathcal{P}(h^X)\, \Omega_X.$$

If $\mathcal{P} \equiv 1$ then $I^{\mathcal{P}}(M \cap gN) = \mathrm{Vol}(M \cap gN)$. In this case the integral (1) is evaluated for all G, K, M and N.

For $\mathcal{P}$ of higher degree the integral (1) is evaluated when G is unimodular and G is transitive on the set on tangent spaces of each of M and N. Then, given $\mathcal{P}$, there is a finite set of invariant polynomials $(\mathcal{Q}_\alpha, \mathcal{R}_\alpha)$ (depending only on $\mathcal{P}$) so that for all appropriate M and N

$$\int_G I^{\mathcal{P}}(M \cap gN)\, \Omega_G(g) = \sum_\alpha I^{\mathcal{Q}_\alpha}(M) I^{\mathcal{R}_\alpha}(N). \tag{2}$$

This generalizes the Chern-Federer kinematic formula to arbitrary homogeneous spaces with an invariant Riemannian metric and leads to new formulas even in the case of submanifolds of Euclidean space.

The approach used here also leads to a "transfer principle" that allows integral geometric formulas to be moved between homogeneous spaces that have the same isotropy subgroups. Thus if G/K and G'/K' are homogeneous spaces with both G and G' unimodular and the subgroups K and K' are isotropic equivalent, then any integral geometric formula of the form (2) that holds for submanifolds of G/K also holds for submanifolds of G'/K'. In particular the transfer principle shows that the Chern-Federer holds in all simply connected space forms of constant sectional curvature and not just in Euclidean space.

1991 Mathematics subject classification: 53C65
Key words and phrases: Integral geometry, Kinematic formula, Integral invariants, Crofton formula, Poincaré formula.