BOUNDING THE NUMBER OF LATTICE POINTS NEAR A CONVEX CURVE BY CURVATURE

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Abstract. We prove explicit bounds on the number of lattice points on or near a convex curve in terms of geometric invariants such as length, curvature, and affine arclength. In several of our results we obtain the best possible constants. Our estimates hold for lattices more general than the usual lattice of integral points in the plane.

1. Introduction

Our goal in this paper is to give explicit and as sharp as possible bounds on the number of lattice points on or near a convex curve in terms of geometric invariants such as length, curvature, and affine arclength for lattices more general than the usual lattice of integral points in the plane.

Definition 1.1. Let $v_0, v_1, v_2 \in \mathbb{R}^2$ be vectors with v_1 and v_2 linearly independent. Then, the ***lattice generated by*** v_1 ***and*** v_2 ***with origin*** v_0 is

$$\mathcal{L} = \mathcal{L}(v_0, v_1, v_2) = \{v_0 + mv_1 + nv_2 : m, n \in \mathbb{Z}\}.$$

Note that the elements of such a lattice need not have integral or even rational components. An invariant of a lattice is the area spanned by v_1 and v_2

$$A_{\mathcal{L}} := |v_1 \wedge v_2|$$

where $v_1 \wedge v_2$ is the determinant of the of the 2×2 matrix with columns v_1 and v_2.

If $\mathcal{C}$ is a curve of differentiability class C^2 and whose curvature κ is positive, then the ***total curvature*** of $\mathcal{C}$ is

$$\boldsymbol{\tau}(\mathcal{C}) := \int_{\mathcal{C}} \kappa \, ds$$

where s is arclength along $\mathcal{C}$ and the ***radius of curvature*** of $\mathcal{C}$ is $\rho = 1/\kappa$. The following are representative of our results.

Theorem 1.2. *Let $\mathcal{C}$ be a C^2 curve with total curvature at most π and whose radius of curvature has a lower bound $\rho \geq R$ for some positive constant R. Let $\mathcal{L}$ be a lattice with*

$$\text{Length}(\mathcal{C}) \leq 2(A_{\mathcal{L}} R)^{1/3}.$$

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Then, $\mathcal{C}$ contains at most two points of $\mathcal{L}$.

This generalizes a theorem of Schinzel (whose proof first appeared in the paper [10, Lemma 2] of Zygmund) where $\mathcal{C}$ is an arc of a circle and the lattice is $\mathbb{Z}^2$. In [2] Cilleruelo shows that when $\mathcal{C}$ is an arc of a circle centered at the origin the sharp form of this inequality has the constant 2 replaced by $2\sqrt[3]{2}$. In our result, with more general lattices and more general curves, the constant 2 is the best possible (see Remark 5.4 below.)

Theorem 1.3. *Let $\mathcal{C}$ be a C^2 curve with total curvature $\boldsymbol{\tau}(\mathcal{C}) = \int_{\mathcal{C}} \kappa\, ds \leq \pi$ and whose radius of curvature satisfies $\rho \geq R_1$ for some $R_1 > 0$. Then for any lattice $\mathcal{L}$*

$$\#(\mathcal{C} \cap \mathcal{L}) < 2 + \frac{\operatorname{Length}(\mathcal{C})}{(A_{\mathcal{L}} R_1)^{1/3}}$$

If also $\rho \leq R_2$, then

$$\#(\mathcal{C} \cap \mathcal{L}) \leq 2 + \left(\frac{R_2\, \boldsymbol{\tau}(\mathcal{C})}{A_{\mathcal{L}} R_1}\right)^{1/3} \operatorname{Length}(\mathcal{C})^{2/3}$$

This result is close to optimal:

Theorem 1.4. *Let $\mathcal{L}$ be a lattice and $n \geq 2$ an integer. There is a convex curve $\mathcal{C}$ that contains exactly n points of $\mathcal{L}$, and lower and upper bounds*

$$R_1 = \min_{P \in \mathcal{C}} \rho(P), \qquad R_2 = \max_{P \in \mathcal{C}} \rho(P)$$

for the radius of curvature of $\mathcal{C}$, so that both the inequalities

$$\frac{\operatorname{Length}(\mathcal{C})}{(R_1 A_{\mathcal{L}})^{1/3}} < n + 2, \qquad \left(\frac{R_2\, \boldsymbol{\tau}(\mathcal{C})}{A_{\mathcal{L}} R_1}\right)^{1/3} \operatorname{Length}(\mathcal{C})^{2/3} < n + 2 \tag{1.1}$$

hold.

The foundational result in this subject is the 1926 paper, [8], of Jarník who proved that the number of integer points on a strictly convex closed curve of length $L > 3$ does not exceed $3(2\pi)^{-1/3} L^{2/3} + O\left(L^{1/3}\right)$ and the exponent and the constant of the leading term are best possible. Therefore, the exponent $2/3$ in Theorem 1.3 is as good as can be expected.

Using that the affine image of a lattice is a lattice, that every ellipse is the affine image of a circle, and that affine arclength (defined in Section 5) is also invariant under affine maps we can transfer results about circles to results about ellipses. One such result is

Theorem 1.5. *Let $\mathcal{C}$ be an arc on an ellipse with affine arclength $\operatorname{Aff}(\mathcal{C})$. Then for any lattice $\mathcal{L}$*

$$\#(\mathcal{C} \cap \mathcal{L}) \leq 2 + \frac{\operatorname{Aff}(\mathcal{C})}{A_{\mathcal{L}}^{1/3}}.$$

We can also estimate the number of points close to a lattice. This involves another invariant of a lattice $\mathcal{L}$, the minimum distance between any two of its points

$$d_{\mathcal{L}} = \min\{\|P - Q\| : P, Q \in \mathcal{L} \text{ and } P \neq Q\}.$$

Theorem 1.6. *Let $\mathcal{C}$ be a convex arc with total curvature at most π with radius of curvature bounded by $R_1 \leq \rho \leq R_2$. Let $\mathcal{L}$ be a lattice and $\delta > 0$ with*

$$\delta < \min\left\{ R_1,\ \frac{d_{\mathcal{L}}^2}{2(R_2 + d_{\mathcal{L}} + \sqrt{(R_2 + d_{\mathcal{L}})^2 - d_{\mathcal{L}}^2})} \right\}$$

and

$$\frac{A_{\mathcal{L}}}{2} - L\delta - \frac{3}{2}\delta^2 > 0.$$

Then,

$$\#\{Q \in \mathcal{L} : \operatorname{dist}(\mathcal{C}, Q) < \delta\} < 2 + \frac{L}{\left(R_1(A_{\mathcal{L}} - 2L\delta - 3\delta^2)\right)^{1/3}}$$

where $L = \text{Length}(\mathcal{C})$.

Theorems estimating the number of lattice points *close* to a curve are more recent. In 1974 Swinnerton-Dyer improved the exponent in Jarník's result for curves which are dilations of a fixed convex C^3 curve. In 1989 Huxley [7] obtained upper bounds for the number of lattice points close to the curve $y = f(x)$, $x \in [M, 2M]$ assuming f satisfies certain smoothness conditions. In particular, Huxley generalized Swinnerton-Dyer's result. A number of papers containing new upper bounds for the number of lattice points close to a curve and applications to different arithmetic functions ensued. For survey of such estimates and their applications see the papers [5] and [6].

Most of our results are based on some new results on the differential geometry of plane convex curves which are of interest on their own right.

Theorem 1.7. *Let $\mathcal{C}$ be a C^2 curve with positive curvature and total curvature $\int_{\mathcal{C}} \kappa\, ds \leq \pi$. If $\mathcal{C}$ intersects a circle of radius R in at least 3 points, then there is a point on $\mathcal{C}$ with $\kappa = 1/R$.*

The structure of this paper is as follows.

Section 2 gives basic facts about lattices and affine maps.

Section 3 contains basic estimates we will be using. The proofs here own a lot to the ideas in the paper [3] of Cilleruelo and Granville.

Section 4 has the proofs of the differential geometric results we require.

Section 5 starts with results about the number of points on a circular arc that are on a general lattice $\mathcal{L}$. Then the affine invariance of the collection of lattices and affine arclength under affine maps is used to transfer these results to the case of lattice points on an arc of an ellipse. The results are new even in the case of the lattice $\mathcal{L} = \mathbb{Z}^2$.

Section 6 has estimates on the number of points of a lattice $\mathcal{L}$ on a convex curve in terms of $A_{\mathcal{L}}$ and bounds on the length and curvature of the curve.

Section 7 contains estimates on the number of points of a lattice $\mathcal{L}$ within δ of a convex arc in terms of $A_{\mathcal{L}}$, $d_{\mathcal{L}}$, and bounds on the length and curvature of the curve.

In Section 8 we show that two of our results are close to being sharp.

2. Lattices and affine maps.

Definition 2.1. An ***affine map*** $\phi\colon \mathbb{R}^n \to \mathbb{R}^n$ is a map of the form

$$\phi(v) = Mv + b$$

where M is a non-singular linear map. Define

$$\det(\phi) = \det(M).$$

The set of lattices is invariant under affine maps.

Proposition 2.2. *Let $\phi\colon \mathbb{R}^2 \to \mathbb{R}^2$ be the affine map*

$$\phi(v) = Mv + b.$$

Then, the image of the lattice $\mathcal{L}(v_0, v_1, v_2)$ under ϕ is

$$\phi\big[\mathcal{L}(v_0, v_1, v_2)\big] = \mathcal{L}(\phi(v_0), Mv_1, Mv_2))$$

and if $\mathcal{L} = \mathcal{L}(v_0, v_1, v_2)$ and $\mathcal{L}^* = \phi\big[\mathcal{L}\big]$ is its image then

$$A_{\mathcal{L}^*} = |\det(\phi)| A_{\mathcal{L}}.$$

This is straightforward and the proof is left to the reader.

Proposition 2.3. *Let P_0, P_1, and P_2 be three non-collinear points of $\mathcal{L} = \mathcal{L}(v_0, v_1, v_2)$. Then, the area of the triangle $\triangle P_0P_1P_2$ is an integral multiple of $A_{\mathcal{L}}/2$ and therefore*

$$\operatorname{Area}(\triangle P_0P_1P_2) \geq \frac{1}{2} A_{\mathcal{L}}.$$

Proof. By the definition of the lattice $\mathcal{L}$ there are integers m_j, n_j with $0 \leq j \leq 2$ so that

$$P_j = v_0 + m_j v_1 + n_j v_2.$$

Since translation does not change areas, we can assume $P_0 = v_0$. Then, the area of $\triangle P_0P_1P_2$ is

$$\begin{aligned}\operatorname{Area}(\triangle P_0P_1P_2) &= \frac{1}{2}|(P_1 - P_0) \wedge (P_2 - P_0)| \\ &= \frac{1}{2}|(m_1v_1 + n_1v_2) \wedge (m_2v_1 + n_2v_2)| \\ &= |m_1n_2 - m_2n_1| \frac{A_{\mathcal{L}}}{2} \\ &\geq \frac{A_{\mathcal{L}}}{2}\end{aligned}$$

as $|m_1n_2 - m_2n_1| \geq 1$ because it is an integer. □

3. Conventions and basic geometric estimates.

All our curves will be of the differentiable class C^2 with nonvanishing first and second derivative vectors. If the orientation (direction of increasing parameter) of a curve is is reversed, it changes the sign of the curvature. As curves with nonvanishing second derivative have nonvanishing curvature, by possibly changing the orientation of the curve, we can, and do, assume all our curves have positive curvature. If we have a finite set of points $\mathcal{F}$ on $\mathcal{C}$, for example if $\#\mathcal{F} = n$, then we order the points $\mathcal{F} = \{P_1, P_2, \ldots, P_n\}$ in the order given by the orientation of the curve. This implies that P_{j+1} is between P_j and P_{j+2}.

Proposition 3.1. *If $\triangle P_0P_1P_2$ is a triangle and its vertices P_0, P_1, and P_2 are on a circle $\mathcal{C}$ of radius R, then the area of the triangle is*

$$\text{Area}(\triangle P_0P_1P_2) = \frac{abc}{4R}$$

where a, b and c are the side lengths of the triangle. Also, the area satisfies the inequality

$$\text{Area}(\triangle P_0P_1P_2) < \frac{(a+b)^3}{16R}.$$

Proof. The formula for the area is a result attributed to Heron of Alexandria [4, Eq. 1.54 p. 13]. To prove the inequality, note $c < a + b$ as a, b and c are the side lengths of a triangle. By the Arithmetic-Geometric mean inequality, $ab \le (a+b)^2/4$ and therefore

$$\text{Area}(\triangle P_0P_1P_2) = \frac{abc}{4R} < \frac{\left((a+b)^2/4\right)(a+b)}{4R} = \frac{(a+b)^3}{16R}.$$

□

The next two results are generalizations of results of Cilleruelo and Granville [3] from circular arcs to more general curves. The proofs are basically axiomatizations of their arguments.

Theorem 3.2 (Basic estimate for closed curves)**.** *Let $\mathcal{C}$ be a closed curve and $P_1, P_2, \ldots, P_N$ points on $\mathcal{C}$ listed in cyclic order around $\mathcal{C}$ with the convention $P_{N+1} = P_1$ and $P_{N+2} = P_2$. Assume there are positive constants A_0 and R_0 such that*

(a) For all j

$$\frac{A_0}{2} \le \text{Area}(\triangle P_jP_{j+1}P_{j+2})$$

(b) For each $j \in \{1, 2, \ldots, N\}$ the points P_j, P_{j+1}, and P_{j+2} are on a circle of radius $\ge R_0$.

Then

$$N < \frac{\text{Length}(\mathcal{C})}{(A_0R_0)^{1/3}}.$$

Theorem 3.3 (Basic estimate for open curves)**.** *Let $\mathcal{C}$ be an immersed curve and $P_1, P_2, \dots, P_N$ points on $\mathcal{C}$ listed in order along $\mathcal{C}$. Assume there are positive constants A_0 and R_0 such that*

(a) For all j

$$\frac{A_0}{2} \le \operatorname{Area}(\triangle P_j P_{j+1} P_{j+2})$$

(b) For each j with $1 \le j \le N-2$ the points P_j, P_{j+1}, and P_{j+2} are on a circle of radius $\ge R_0$.

Then

$$N < 2 + \frac{\operatorname{Length}(\mathcal{C})}{(A_0 R_0)^{1/3}}.$$

Proof of Theorem 3.2. Let R_j be the radius of the circle through P_j, P_{j+1}, and P_{j+2}. To simplify notation, we set

$$a_j := \|P_{j+1} - P_j\|.$$

Then by Proposition 3.1 and using $R_j \ge R_0$

$$\begin{aligned} \operatorname{Area}(\triangle P_j P_{j+1} P_{j+2}) &< \frac{(a_j + a_{j+1})^3}{16 R_j} \\ &\le \frac{\big(a_j + a_{j+1}\big)^3}{16 R_0} \end{aligned}$$

Combining this with assumption (a) gives

$$1 < \frac{(a_j + a_{j+1})^3}{8 A_0 R_0}.$$

Take cube roots

$$1 < \frac{a_j + a_{j+1}}{2(A_0 R_0)^{1/3}}$$

and sum on j

$$N < \sum_{j=1}^{N} \frac{a_j + a_{j+1}}{2(A_0 R_0)^{1/3}} = \frac{1}{(A_0 R_0)^{1/3}} \sum_{j=1}^{N} a_j,$$

where we have used $\sum_{j=1}^{N} a_{j+1} = \sum_{j=1}^{N} a_j$. This sum is the length of a polygon inscribed in $\mathcal{C}$ and thus $\sum_{j=1}^{N} a_j \le \operatorname{Length}(\mathcal{C})$ which completes the proof. □

Proof of Theorem 3.3. As in the proof of Theorem 3.2 we have

$$1 < \frac{a_j + a_{j+1}}{2(A_0 R_0)^{1/3}}$$

but this time only holding for $1 \le j \le N-2$. Sum on this to get

$$\begin{aligned} N-2 &< \sum_{j=1}^{N-2} \frac{a_j + a_{j+1}}{2(A_0R_0)^{1/3}} = \frac{1}{2(A_0R_0)^{1/3}} \left(\sum_{j=1}^{N-2} a_j + \sum_{j=1}^{N-2} a_{j+1} \right) \\ &< \frac{1}{(A_0R_0)^{1/3}} \sum_{j=1}^{N-1} a_j \le \frac{\operatorname{Length}(\mathcal{C})}{(A_0R_0)^{1/3}}. \end{aligned}$$

□

4. Some differential geometry.

Let $\mathcal{C}$ be a C^2 plane curve and let $\gamma\colon [a,b] \to \mathbb{R}^2$ be a unit speed, that is $\|\gamma'(s)\| = 1$ for all s, parametrization of $\mathcal{C}$. Let $\mathbf{t}(s) = \gamma'(s)$ be the unit tangent and $\mathbf{n}(s)$ the unit normal where we choose $\mathbf{n}$ to be $\mathbf{t}$ rotated by $\pi/2$ in the positive direction. Then the ***curvature*** function along $\mathcal{C}$ is defined by

$$\frac{d\mathbf{t}}{ds} = \kappa(s)\mathbf{n}.$$

As remarked above, we orient all our curves so that the curvature is positive.

There is another way to define curvature which will be useful to us. As $\mathbf{t}(s)$ is a unit vector, it can be written as

$$\mathbf{t} = (\cos(\theta(s)), \sin(\theta(s)))$$

where θ is a C^1 function and is the angle the tangent makes with the positive x-axis. Then

$$\frac{d\mathbf{t}}{ds} = \frac{d\theta}{ds}(-\sin(\theta(s)), \cos(\theta(s))) = \kappa(s)\mathbf{n}(s).$$

Therefore, the curvature is the rate of change of the angle with respect to arclength:

$$\kappa = \frac{d\theta}{ds}.$$

The ***total curvature*** of $\mathcal{C}$ is the integral of curvature with respect to arclength and is the total change in the angle of the tangent vector:

$$\boldsymbol{\tau}(\mathcal{C}) := \int_{\mathcal{C}} \kappa\, ds = \int_a^b \frac{d\theta}{ds}\, ds = \theta(b) - \theta(a).$$

This interpretation makes it easy to compare the total curvature of two curves with the same endpoints.

Proposition 4.1. *Let $\mathcal{C}_1$ and $\mathcal{C}_2$ be convex curves with the same endpoints and with $\mathcal{C}_1$ inside $\mathcal{C}_2$ in the sense that $\mathcal{C}_1$ is inside the convex hull of $\mathcal{C}_2$ (see Figure 1). Then the total curvature of $\mathcal{C}_1$ is less than or equal to the total curvature of $\mathcal{C}_2$:*

$$\int_{\mathcal{C}_1} \kappa_{\mathcal{C}_1}\, ds \le \int_{\mathcal{C}_2} \kappa_{\mathcal{C}_2}\, ds.$$

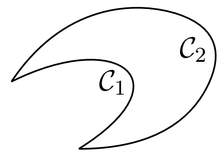

Figure 1. The total curvature of $\mathcal{C}_2$ is greater than the total curvature of $\mathcal{C}_1$.

Proof. This is obvious from Figure 1 and the interpolation of the total curvature as the change in angle along the curve. The reader wanting a more detailed (or more highbrow) proof can construct one from the Gauss-Bonnet formula for surfaces with boundaries having canners. For this see Equation (4) in the excellent expository article [1] by S.-S. Chern. □

Various elementary inequalities between bounds on the length, total curvature, and bounds on the radius will come up often enough that it is worth recording them.

Proposition 4.2. *If the radius of curvature of a curve satisfies $R_1 \leq \rho \leq R_2$ for some positive constants R_1 and R_2 and if L is the length of $\mathcal{C}$ then*

$$R_1 \boldsymbol{\tau}(\mathcal{C}) \leq L \leq R_2 \boldsymbol{\tau}(\mathcal{C})$$

and

$$\frac{L}{R_1^{1/3}} \leq \left(\frac{\boldsymbol{\tau}(\mathcal{C})R_2}{R_1}\right)^{1/3} L^{2/3} \leq \frac{\boldsymbol{\tau}(\mathcal{C})R_2}{R_1^{1/3}}.$$

Proof. The first of these follows from $\boldsymbol{\tau}(\mathcal{C}) = \int_{\mathcal{C}} \kappa\, ds = \int_{\mathcal{C}} (1/\rho)\, ds$ and $R_1 \leq \rho \leq R_2$. The second follows from just using $L \leq \boldsymbol{\tau}(\mathcal{C})R_2$

$$\begin{aligned}\frac{L}{R_1^{1/3}} &= \left(\frac{L}{R_1}\right)^{1/3} L^{2/3} \leq \left(\frac{\boldsymbol{\tau}(\mathcal{C})R_2}{R_1}\right)^{1/3} L^{2/3} \\ &\leq \left(\frac{\boldsymbol{\tau}(\mathcal{C})R_2}{R_1}\right)^{1/3} (\boldsymbol{\tau}(\mathcal{C})R_2)^{2/3} = \frac{\boldsymbol{\tau}(\mathcal{C})R_2}{R_1^{1/3}}\end{aligned}$$

□

Another basic tool we will use is an elementary maximum principle. This is well-known, but we include a short proof for completeness.

Proposition 4.3 (Maximum Principle)**.** *Let $\mathcal{C}_1$ and $\mathcal{C}_2$ be convex curves with $\mathcal{C}_1$ inside the convex hull of $\mathcal{C}_2$ and tangent to $\mathcal{C}_2$ at some point P (which could be endpoints of $\mathcal{C}_1$ and $\mathcal{C}_2$). Then at P*

$$\kappa_{\mathcal{C}_1} \geq \kappa_{\mathcal{C}_2}.$$

An informal restatement is that if $\mathcal{C}_1$ is internally tangent to $\mathcal{C}_2$ at P, then $\mathcal{C}_1$ is as least as curved as $\mathcal{C}_2$ at P.

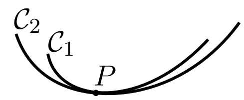

Figure 2. C_1 is at least as curved as $\mathcal{C}_2$ at the point P.

Proof. In an appropriate coordinate system, and possibly working with smaller pieces of the curves near P, we can write $\mathcal{C}_1$ and $\mathcal{C}_2$ as graphs $y = f_1(x)$ and $y = f_2(x)$ respectively. Then the hypothesis of the proposition is that the function $f_1 - f_2$ has a local minimum at P. The first and second derivative tests yield that if $P = (x_0, f(x_0))$, then $f_1'(x_0) - f_2'(x_0) = 0$ and $f_1''(x_0) - f_2''(x_0) \geq 0$. Using this equality and inequality and the standard formula for the curvature of graphs we get

$$\kappa_{\mathcal{C}_1}(P) = \frac{f_1''(x_0)}{(1 + f_1'(x_0)^2)^{3/2}} \geq \frac{f_2''(x_0)}{(1 + f_2'(x_0)^2)^{3/2}} = \kappa_{\mathcal{C}_2}(P).$$

□

Another well known fact is that two C^2 curves with common endpoints, tangent, and curvature can be joined together to form a C^2 curve. Again, we include a short proof.

Lemma 4.4 (Splicing Lemma)**.** *Let $\mathcal{C}_1$ and $\mathcal{C}_2$ be two curves of class C^2 such that the terminal point of $\mathcal{C}_1$ is the initial point of $\mathcal{C}_2$, and that at this common point the two curves have the same tangent and curvature as in Figure 3. Then $\mathcal{C}_1 \cup \mathcal{C}_2$ is a curve of class C^2.*

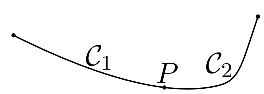

Figure 3. The curves $\mathcal{C}_1$ and $\mathcal{C}_2$ have the same tangent and curvature at P. This implies the union $\mathcal{C} = \mathcal{C}_1 \cup \mathcal{C}_2$ is also a C^2 curve.

Proof. There are coordinates so that near P, $\mathcal{C}_1$ is the graph of a function $y = f_1(x)$ for on the interval $[a, 0]$ and $\mathcal{C}_2$ is the graph of $y = f_2(x)$ on $[0, b]$ for C^2 functions f_1 and f_2. As the terminal point of $\mathcal{C}_1$ is the initial point of $\mathcal{C}_2$ we have $f_1(0) = f_2(0)$. That the curves have the same tangent at this point implies $f_1'(0) = f_2'(0)$. The equality of the curvatures at $x = 0$ gives $f_1''(0)/(1 + f_1'(0)^2)^{3/2} = f_2''(0)/(1 + f_2'(0)^2)^{3/2}$ which implies $f_1''(0) = f_2''(0)$. Therefore, the function given by $f(x) = f_1(x)$ on $[a, 0]$ and $f(x) = f_2(x)$ on $[0, b]$ is continuous with continuous first and second derivatives. Whence $\mathcal{C} = \mathcal{C}_1 \cup \mathcal{C}_2$ is the graph of a C^2 function near the common endpoint, showing that $\mathcal{C}$ is C^2. □

Lemma 4.5. *Let $\mathcal{C}_1$ and $\mathcal{C}_2$ be C^2 convex curves with the same endpoints and with $\mathcal{C}_1$ contained in the convex hull of $\mathcal{C}_2$. Assume the total curvature of $\mathcal{C}_2$ satisfies*

$$\int_{\mathcal{C}_2} \kappa\, ds \leq \pi.$$

Then $\mathcal{C}_2$ is at least as curved as $\mathcal{C}_1$ in the sense that

$$\max_{P\in\mathcal{C}_2} \kappa_{\mathcal{C}_2}(P) \geq \min_{Q\in\mathcal{C}_1} \kappa_{\mathcal{C}_1}(Q).$$

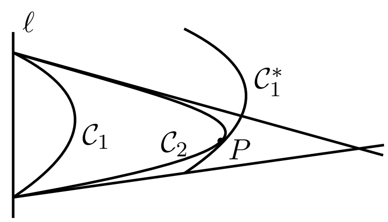

Figure 4. $\mathcal{C}_1$ can be translated to a position $\mathcal{C}_1^*$ where it is externally tangent to $\mathcal{C}_2$ at the point P. At P the curve $\mathcal{C}_2$ is at least as curved as $\mathcal{C}_1^*$.

Proof. Let ℓ be the line through the common endpoints of the two curves and consider the tangent lines to $\mathcal{C}_2$ at its endpoints as in Figure 4. Because the total curvature of $\mathcal{C}_2$ is at most π these lines will either be be parallel (when the total curvature is π) or will intersect on the same side of ℓ as $\mathcal{C}_1$ and $\mathcal{C}_2$. Translate $\mathcal{C}_1$ keeping one of its endpoints on one of the tangent lines to a position $\mathcal{C}_1^*$ where it is tangent to $\mathcal{C}_2$ at a point P (this is the farthest translated position where $\mathcal{C}_2$ and $\mathcal{C}_1^*$ still intersect). By the maximum principle $\kappa_{\mathcal{C}_2}(P) \geq \kappa_{\mathcal{C}_1^*}(P)$. As translation preserves curvature this completes the proof. □

The hypothesis $\int_{\mathcal{C}_2} \kappa\, ds \leq \pi$ is necessary as can be seen in the example of the two circular arcs in Figure 5.

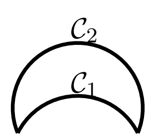

Figure 5. The curvature of $\mathcal{C}_2$ is everywhere less than the curvature of $\mathcal{C}_1$.

Theorem 4.6. *Let $\mathcal{C}_1$ and $\mathcal{C}_2$ be C^2 closed convex curves that intersect in three or more points. Then, they have comparable curvature in the sense that there are points P on $\mathcal{C}_1$ and Q on $\mathcal{C}_2$ with $\kappa_{\mathcal{C}_1}(P) = \kappa_{\mathcal{C}_2}(Q)$.*

Corollary 4.7. *Let $\mathcal{C}$ be a closed convex C^2 curve that intersects a circle of radius R in three or more points. Then, there is a point on $\mathcal{C}$ with $\kappa = 1/R$.*

Proof of Theorem 4.6. If $\mathcal{C}_1$ and $\mathcal{C}_2$ intersect in infinitely many points, then let P be an accumulation point of the set of intersection points. At P the two curves will have contact of order at least 2 (and in fact, infinite order if the curves are of class C^∞) and therefore, have the same curvature at P. Thus, we can assume the two curves only intersect in finitely many points.

Claim. *There are points P_j on $\mathcal{C}_j$ for $j = 1, 2$ such that $\kappa_{\mathcal{C}_1}(P_1) \le \kappa_{\mathcal{C}_2}(P_2)$.*

Assuming the claim the theorem follows. For the claim implies the function $\kappa_{\mathcal{C}_2} - \kappa_{\mathcal{C}_1}$ is non-negative at some point on the Cartesian product $\mathcal{C}_1 \times \mathcal{C}_2$. By symmetry this function is also non-positive at some point. As $\mathcal{C}_1 \times \mathcal{C}_2$ is connected this implies $\kappa_{\mathcal{C}_2} - \kappa_{\mathcal{C}_1} = 0$ at some point, which is equivalent to the conclusion of the theorem.

The proof of the claim splits into three cases.

Case 1: $\mathcal{C}_1$ is externally tangent to $\mathcal{C}_2$ at some point of intersection. Then the claim follows directly from the maximum principle (Proposition 4.3).

Case 2: $\mathcal{C}_1$ is internally tangent to $\mathcal{C}_2$ at some point of intersection. Let $\mathcal{C}_1$ be internally tangent to $\mathcal{C}_2$ at the point P. Let P_- and P_+ be the points of intersection that are on either side of P (these exist as there are only finitely many points of intersection). As the total curvature of $\mathcal{C}_2$ is 2π, at least on of the two arcs $\mathcal{C}_2\big|_{P_-}^{P}$ or $\mathcal{C}_2\big|_{P}^{P_+}$ will have total curvature $\le \pi$. Then Lemma 4.5 implies the conclusion of the claim holds.

Case 3: At every point of intersection $\mathcal{C}_1$ crosses $\mathcal{C}_2$. Between each two consecutive points of intersection the arc of $\mathcal{C}_2$ between these points is either inside of $\mathcal{C}_1$, call such arcs *positive*, or outside of $\mathcal{C}_1$, call such arcs *negative*. In the current case each point of intersection is between a positive and negative arc of $\mathcal{C}_2$. Therefore, the total number of points of intersection is even and the number of positive arcs of $\mathcal{C}_2$ is half of this number. The number of points of intersection is at least 3 and therefore the $\mathcal{C}_2$ has at least two positive arcs. And again, as the total curvature of $\mathcal{C}_2$ is 2π at least one of these arcs has total curvature $\le \pi$, and again we can use Lemma 4.5 to see the claim holds. □

Theorem 4.8. *Let $\mathcal{C}$ be a C^2 convex curve with total curvature satisfying $\int_{\mathcal{C}} \kappa\, ds \le \pi$ that intersects a circle of radius R in three or more points. Then, there is a point on $\mathcal{C}$ with curvature $\kappa = 1/R$.*

Proof. We first consider the case when $\int_{\mathcal{C}} \kappa\, ds < \pi$. Let P_0 be the initial point of $\mathcal{C}$ and P_1 the terminal point.

Let κ_0 be the curvature of $\mathcal{C}$ at P_0 and κ_1 its curvature at P_1. Let $\alpha = \pi - \int_{\mathcal{C}} \kappa\, ds$. Construct a curve $\mathcal{C}_1$ with total curvature α and with curvature κ_1 at its initial point and κ_0 at its terminal point and with its curvature everywhere between κ_0 and κ_1. As explicit example of such a curve can be constructed by letting $\theta\colon [0, \alpha] \to \mathbb{R}$ be a function with derivative

$$\theta'(t) = \frac{\alpha - t}{\alpha}\kappa_1 + \frac{t}{\alpha}\kappa_0.$$

and letting

$$\gamma(s) = \int_0^s (\cos\theta(t), \sin\theta(t))\, dt.$$

Then γ is unit speed curve with curvature $\kappa(s) = \theta'(s)$. By rotating and translating $\mathcal{C}_1$ we can move it until its initial point is P_1 and $\mathcal{C}$ and $\mathcal{C}_1$ have the same tangent vector at P_1 as in Figure 6.

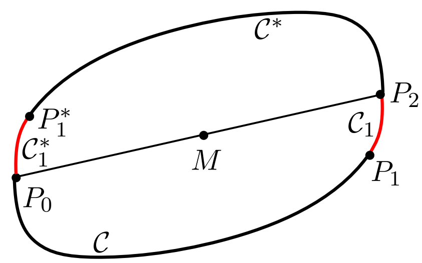

Figure 6. Extend the curve $\mathcal{C}$ by the curve $\mathcal{C}_1$ so that that total curvature of $\mathcal{C} \cup \mathcal{C}_1$ is π and is of class C^2. Let P_2 be the terminal point of this union. Rotate these curves around the midpoint, M, of the segment between P_0 and P_2. The resulting closed curve will be of class C^2 as can be seen by four applications of The Splicing Lemma 4.4.

Take the resulting curve $\mathcal{C} \cup \mathcal{C}_1$ and rotate it about the midpoint, M, of the segment between P_0 and P_1 and let $\mathcal{C}^*$ and $\mathcal{C}_1^*$. Then the union $\mathcal{B} = \mathcal{C} \cup \mathcal{C}_1 \cup \mathcal{C}^* \cup \mathcal{C}_1^*$ is a closed convex curve. As $\mathcal{C}$ and $\mathcal{C}_1$ are C^2 the curve $\mathcal{B}$ is C^2 except possibly at the points P_0, P_1, P_2, and P_1^*. At P_1 the curves $\mathcal{C}$ and $\mathcal{C}_1$ have the same tangent vector and by construction they have the same curvature at P_1. Therefore $\mathcal{B}$ is C^2 in a neighborhood of P_1 by the Spicing Lemma 4.4. A similar argument shows $\mathcal{B}$ is C^2 near the remaining points P_0, P_2, and P_1^*.

As $\mathcal{C}$ intersects some circle of radius R in three or more points, the curve $\mathcal{B}$ will also meet this circle in three or more points. By Corollary 4.7 the curve $\mathcal{B}$ contains a point P where $\kappa = 1/R$. If P is on $\mathcal{C}$ we are done. It P is on $\mathcal{C}^*$, then, as $\mathcal{C}^*$ is just a rotation of $\mathcal{C}$, there is a point of $\mathcal{C}$ with $\kappa = 1/R$. If P is on $\mathcal{C}_1$, then by the construction of $\mathcal{C}_1$ we have $\kappa_{\mathcal{C}_1}(P) = 1/R$ is between κ_0 and κ_1 and by the intermediate value theorem there is a point of $\mathcal{C}$ with curvature $1/R$. A similar argument works in the case when P is on $\mathcal{C}_1^*$. This covers all the cases and completes the proof in the case the total curvature of $\mathcal{C}$ is less than π.

If the total curvature of $\mathcal{C}$ is π and $\mathcal{C}$ intersects the circle of radius R in four or more points, then it will have proper sub-arc that intersects the circle in three or more points and such that this sub-arc will have total curvature less than π and we are back in the case we have just covered. So, assume $\mathcal{C}$ intersects the circle of radius R in exactly three points. If one of the endpoints of $\mathcal{C}$ is not a point of intersection, then there is again a proper sub-arc of $\mathcal{C}$ that contains the three points of intersection with the circle and this sub-arc will have total curvature less than π and we are done.

Therefore, we can assume that $\mathcal{C}$ intersects the circle of radius R in exactly three points P_0, P_1, and P_2 and that P_0 and P_2 are endpoints of $\mathcal{C}$. As the total curvature of $\mathcal{C}$ is π the tangent lines to $\mathcal{C}$ at P_0 and P_2 are parallel. By a rotation we can assume these are vertical and that $\mathcal{C}$ is the graph of a convex function. The proof now splits into cases. Let $\mathcal{S}$ be the circle of radius R intersecting $\mathcal{C}$ in the points P_0, P_1, and P_2.

Case 1: The points P_0, P_1, and P_2 are all on the closed lower half of $\mathcal{S}$.

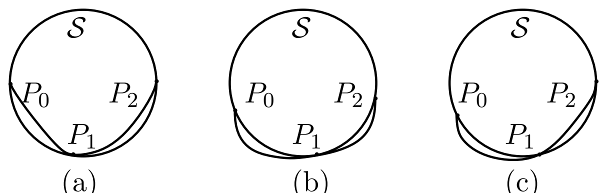

Figure 7. The three cases where the endpoints of $\mathcal{C}$ are on the closed lower half of the circle $\mathcal{S}$.

There are three sub-cases. First, $\mathcal{C}$ could be internally tangent to $\mathcal{S}$ at P_1 as in Figure 7 (a). Then, by the Maximum Principle $\kappa_{\mathcal{C}}(P_1) \geq 1/R$. The total curvature of the lower half circle is π and thus by Lemma 4.5 there is a point Q of $\mathcal{C}$ between P_0 and P_1 with $\kappa_{\mathcal{C}}(Q) \leq 1/R$. Thus, there is a point on $\mathcal{C}$ with curvature $1/R$.

Sub-case (b) is as in Figure 7 (b) where $\mathcal{C}$ is externally tangent to $\mathcal{S}$ at P_1. By the Maximum Principle $\kappa_{\mathcal{C}}(P_1) \leq 1/R$, and as the total curvature of $\mathcal{C}$ is π Lemma 4.5 gives a point between P_0 and P_1 where $\kappa_{\mathcal{C}} \geq 1/R$. Thus, there is a point with $\kappa_{\mathcal{C}} = 1/R$.

Sub-case (c) is as in Figure 7 (c) where $\mathcal{C}$ crosses $\mathcal{S}$ at P_1. Then $\mathcal{C}$ and the lower half of the circle have total curvature π and therefore Lemma 4.5 can be applied twice, once between P_0 and P_1 to find a point of $\mathcal{C}$ with $\kappa_{\mathcal{C}} \geq 1/R$, and once between P_1 and P_2 to find a point on $\mathcal{C}$ with $\kappa_{\mathcal{C}} \leq 1/R$. So again, there is a point with $\kappa_{\mathcal{C}} = 1/R$.

Case 2: At least one of the endpoints of $\mathcal{C}$ is in the open upper half of the circle $\mathcal{S}$.

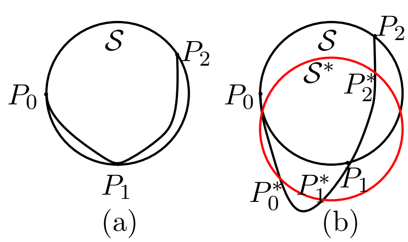

Figure 8. The two cases where one endpoint of C is on the open upper half of the circle $\mathcal{S}$.

Let P_2 be an endpoint that is in the upper half of $\mathcal{S}$. As the tangent to $\mathcal{C}$ at P_2 is vertical the curve $\mathcal{C}$ will contain points in the interior of $\mathcal{S}$. Thus, there are two sub-cases.

Sub-case (a) is when $\mathcal{C}$ is internally tangent to $\mathcal{S}$ at P_1 as in Figure 8 (a). By the maximum principle $\kappa_{\mathcal{C}} \geq 1/R$ at P_1. The total curvature of the circle is 2π and therefore at least one of the arc $\mathcal{S}\big|_{P_0}^{P_1}$ or $\mathcal{S}\big|_{P_1}^{P_2}$ has total curvature $\leq \pi$. Lemma 4.5 then gives a point of $\mathcal{C}$ with $\kappa_{\mathcal{C}} \leq 1/R$ and there is a point with $\kappa_{\mathcal{C}} = 1/R$.

Sub-case (b) is when $\mathcal{C}$ crosses $\mathcal{S}$ at P_1. As the tangent to $\mathcal{C}$ at P_2 is vertical the part of $\mathcal{C}$ near P_2 is interior to $\mathcal{S}$. Translate $\mathcal{S}$ downward to a position $\mathcal{S}^*$ so that it contains points in the region bounded by the two curves $\mathcal{S}\big|_{P_0}^{P_1}$ and $\mathcal{C}\big|_{P_0}^{P_1}$. As the lower half of $\mathcal{S}^*$ contains both points inside and outside this region and it does not intersect $\mathcal{S}$, we see the lower half of $\mathcal{S}^*$ intersects $\mathcal{C}\big|_{P_0}^{P_1}$ in at least two points P_0^* and P_1^*. As P_1 is inside of $\mathcal{S}^*$ and P_2 is outside of $\mathcal{S}$ the circle $\mathcal{S}^*$ will intersect $\mathcal{S}^*$ at some point P_2^* on $\mathcal{C}\big|_{P_1}^{P_2}$. Therefore $\mathcal{C}\big|_{P_0^*}^{P_2^*}$ intersects the circle $\mathcal{S}^*$ of radius R in at least three points and as it is a proper sub-arc of $\mathcal{C}$ it has total curvature $< \pi$. Therefore $\mathcal{C}\big|_{P_0^*}^{P_2^*}$, and thus also $\mathcal{C}$, has a point with curvature $= 1/R$. $\square$

The curves in Figure 9 show the hypothesis $\int_{\mathcal{C}} \kappa\, ds \leq \pi$ in Theorem 4.8 is best possible.

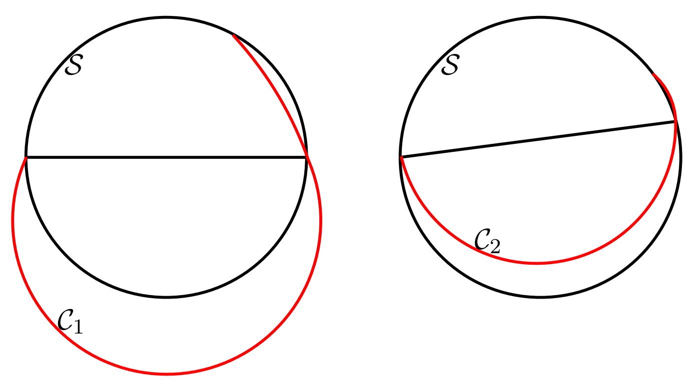

Figure 9. The circle $\mathcal{S}$ has radius R. In the figure on the left, $\mathcal{C}_1$ meets $\mathcal{S}$ in three points and has curvature $< 1/R$ at all points. In the figure on the right, $\mathcal{C}_2$ meets $\mathcal{S}$ in three points and has curvature $> 1/R$ at all points. The total curvatures of $\mathcal{C}_1$ and $\mathcal{C}_2$ can be made arbitrarily close to π.

5. Affine arclength and bounding the number of lattice points on circles and ellipses.

Theorem 5.1. *Let $\mathcal{C}$ be an arc of length L of a circle with radius R and let $\mathcal{L}$ be a lattice. If*

$$\frac{L}{R^{1/3}} \leq 2(A_{\mathcal{L}})^{1/3}$$

then $\mathcal{C}$ contains at most 2 points of the lattice $\mathcal{L}$.

Theorem 5.2. *Let $\mathcal{C}$ be an arc of length L of a circle with radius R and let $\mathcal{L}$ be a lattice. Then*

$$\#(\mathcal{C} \cap \mathcal{L}) < 2 + \frac{L}{(A_{\mathcal{L}} R)^{1/3}}.$$

Theorem 5.3. *Let $\mathcal{S}$ be a circle of radius R and $\mathcal{L}$ a lattice. Then*

$$\#(\mathcal{S} \cap \mathcal{L}) < \frac{\text{Length}(\mathcal{S})}{(A_{\mathcal{L}} R)^{1/3}} = \frac{2\pi R^{2/3}}{A_{\mathcal{L}}^{1/3}}.$$

Proof of Theorem 5.1. If $\mathcal{C} \cap \mathcal{L}$ has three or more points then let P_0, P_1, and P_2 be distinct points in $\mathcal{C} \cap \mathcal{L}$. Then the triangle $\triangle P_0 P_1 P_2$ has area $\geq \frac{1}{2} A_{\mathcal{L}}$ by Proposition 2.3 and Proposition 3.1 implies

$$\frac{A_{\mathcal{L}}}{2} < \frac{(\|P_1 - P_0\| + \|P_2 - P_1\|)^3}{16R} < \frac{L^3}{16R}$$

which simplifies to $L > 2(A_{\mathcal{L}} R)^{1/3}$. This proves the contrapositive of the theorem. □

Proof of Theorem 5.2. Let $P_1, P_1, \ldots, P_N$ be the points of $\mathcal{C} \cap \mathcal{L}$. Then Proposition 2.3 implies the hypothesis of Theorem 3.3 holds with $A_0 = A_{\mathcal{L}}$ and $R_0 = R$. □

Proof of Theorem 5.3. This proof is identical to the previous proof except that this time Theorem 3.2 rather than Theorem 3.3 is used. □

Remark 5.4. The constant 2 in Theorem 5.1 is sharp. Let $\mathcal{S}$ be a circle of radius R and let P_0, P_1, and P_2 be three points on $\mathcal{S}$ with the arclength from P_0 to P_1 and the arclength from P_1 to P_2 being $L/2$ as in Figure 10.

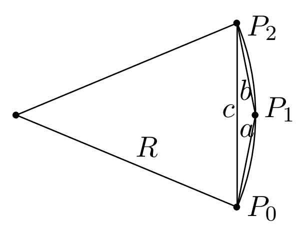

Figure 10. The points P_0, P_1, and P_2 are on a circle of radius R, the arclength between P_0 and P_2 is L and P_1 is the midpoint of the arc between P_0 and P_1. The lengths, a, b, and c of the sides of $\triangle P_0 P_1 P_2$ are as shown.

Recalling if two points P and Q on a circle of radius R are the endpoints of an arc of length λ on the circle, then $\|P - Q\| = 2R\sin(\lambda/(2R))$ we find that the side lengths of $\triangle P_0P_1P_2$ are given by

$$a(R) = b(R) = 2R\sin\left(\frac{L}{4R}\right), \qquad c(R) = 2R\sin\left(\frac{L}{2R}\right)$$

Let $\mathcal{L}_R = \mathcal{L}(P_0, v_1, v_2)$ where $v_j = P_j - P_0$ for $j = 1, 2$. For this lattice

$$A_{\mathcal{L}_R} = 2\,\mathrm{Area}(\triangle P_0P_1P_2) = \frac{a(R)b(R)c(R)}{2R}$$

where the second equality follows from Proposition 3.1. Using these in the inequality in Theorem 5.1 and doing a bit of algebra gives

$$L \le 2(A_{\mathcal{L}_R})^{1/3} = 2\left(\frac{a(R)b(R)c(R)}{2}\right)^{1/3}.$$

However,

$$\lim_{R\to\infty} a(R) = \lim_{R\to\infty} b(R) = \frac{L}{2}, \qquad \lim_{R\to\infty} c(R) = L.$$

Therefore

$$L \le \lim_{R\to\infty} 2\left(\frac{a(R)b(R)c(R)}{2}\right)^{1/3} = 2\left(\frac{L^3}{8}\right)^{1/3} = L,$$

showing the inequality is sharp.

This example is a bit unsatisfying as we are choosing the lattice to depend on both R and L. A natural question is given a lattice $\mathcal{L}$ what is the best constant $C_{\mathcal{L}}$ such that for any arc of length L on a circle of radius R with

$$L < C_{\mathcal{L}}(A_{\mathcal{L}}R)^{1/3}$$

contains at most 2 points of $\mathcal{L}$. Theorem 5.1 together with these examples shows

$$\inf_{\mathcal{L}} C_{\mathcal{L}} = 2.$$

For the lattice $\mathcal{L} = \mathbb{Z}^2$ and restricting to circles centered at the origin Cilleruelo [2] and Cilleruelo and Granville [3] have shown $C_{\mathbb{Z}^2} = 2(2^{1/3})$. To the best of our knowledge $C_{\mathcal{L}}$ is not known for any other lattice.

We recall the definition of affine arclength. Let $\mathcal{C}$ be a curve with positive curvature and let $\gamma\colon [a, b] \to \mathcal{C}$ be a parametrization of $\mathcal{C}$. Then, the ***affine arclength*** of $\mathcal{C}$ is given by

$$\mathrm{Aff}(\mathcal{C}) = \int_a^b (\gamma'(t) \wedge \gamma''(t))^{1/3}\,dt.$$

If $\phi(v) = Mv + b$ is an affine map with $\det(M) > 0$, then it is straightforward to check that if $c(t) = \phi(\gamma(t))$ then

$$c'(t) \wedge c''(t) = (M\gamma'(t)) \wedge (M\gamma''(t)) = \det(M)\gamma'(t) \wedge \gamma''(t)$$

and therefore, the affine arclength transforms under affine maps with positive determent by the rule

$$\operatorname{Aff}(\phi[\mathcal{C}]) = \det(M)^{1/3} \operatorname{Aff}(\mathcal{C}). \tag{5.1}$$

It $\gamma\colon [a,b] \to \mathbb{R}^2$ is unit speed in the Euclidean sense and has positive curvature then $\gamma'(s) = \mathbf{t}(s)$, and $\gamma''(s) = \kappa(s)\mathbf{n}(s)$. Thus, $\gamma'(s) \wedge \gamma''(s) = \kappa(s)$. This implies:

Lemma 5.5. *Let $\mathcal{C}$ be a C^2 curve with positive curvature κ. Then, the affine arclength of $\mathcal{C}$ is*

$$\operatorname{Aff}(\mathcal{C}) = \int_a^b \kappa^{1/3}\, ds. \tag{5.2}$$

In particular, if $\mathcal{C}$ is an arc on a circle of radius R, then

$$\operatorname{Aff}(\mathcal{C}) = \frac{\operatorname{Length}(\mathcal{C})}{R^{1/3}}.$$

By an ***ellipse*** we mean a curve $\mathcal{E}$ with an equation of the form

$$A(x-x_0)^2 + 2B(x-x_0)(y-y_0) + C(y-y_0)^2 = 1$$

where the matrix $\begin{bmatrix} A & B \\ B & C \end{bmatrix}$ is positive definite. A fact we will use is that if $\mathcal{E}$ is an ellipse, then there is affine map ϕ such that $\det(\phi) = 1$ and the image $\phi\big[\mathcal{E}\big]$ is a circle.

Theorem 5.6. *Let $\mathcal{C}$ be an arc on an ellipse $\mathcal{E}$ and let $\mathcal{L}$ be a lattice such that*

$$\operatorname{Aff}(\mathcal{C}) \le 2(A_{\mathcal{L}})^{1/3}.$$

Then, $\mathcal{C}$ contains at most 2 points of the lattice $\mathcal{L}$.

Theorem 5.7. *If $\mathcal{C}$ is an arc on an ellipse and $\mathcal{L}$ is a lattice, then*

$$\#(\mathcal{C} \cap \mathcal{L}) < 2 + \frac{\operatorname{Aff}(\mathcal{C})}{A_{\mathcal{L}}^{1/3}}.$$

Theorem 5.8. *Let $\mathcal{E}$ be an ellipse and $\mathcal{L}$ a lattice. Then*

$$\#(\mathcal{E} \cap \mathcal{L}) < \frac{\operatorname{Aff}(\mathcal{E})}{A_{\mathcal{L}}^{1/3}}.$$

Proof of Theorem 5.6. Choose an affine map ϕ with $\det(\phi) = 1$ and such that the image $\mathcal{S} := \phi\big[\mathcal{E}\big]$ is a circle and let R be the radius of this circle. Let $\mathcal{C}^* = \phi\big[\mathcal{C}\big]$ and $\mathcal{L}^* = \phi\big[\mathcal{L}\big]$. Then by the invariance property of affine arclength under affine maps, Lemma 5.5, and $A_{\mathcal{L}^*} = \det(M) A_{\mathcal{L}} = A_{\mathcal{L}}$

$$\frac{\operatorname{Length}(\mathcal{C}^*)}{R^{1/3}} = \operatorname{Aff}(\mathcal{C}^*) = \det(M)^{1/3} \operatorname{Aff}(\mathcal{C}) = \operatorname{Aff}(\mathcal{C}) \le (A_{\mathcal{L}})^{1/3} = (A_{\mathcal{L}^*})^{1/3}.$$

Thus by Theorem 5.1 $\#(\mathcal{C}^* \cap \mathcal{L}^*) \le 2$. But ϕ is a bijection so this implies $\#(\mathcal{C} \cap \mathcal{L}) \le 2$. ∎

Proofs of Theorems 5.7 and 5.8. Using the notation of the proof of Theorem 5.6, the invariance properties of affine arclength, and the equalities $\mathrm{Aff}(\mathcal{C}) = \mathrm{Aff}(\mathcal{C}^*) = \mathrm{Length}(\mathcal{C}^*)/R^{1/3}$, $\mathrm{Aff}(\mathcal{E}) = 2\pi R^{2/3}$, $A_{\mathcal{L}} = A_{\mathcal{L}^*}$, $\#(\mathcal{C} \cap \mathcal{L}) = \#(\mathcal{C}^* \cap \mathcal{L}^*)$, and $\#(\mathcal{E} \cap \mathcal{L}) = \#(\mathcal{S} \cap \mathcal{L}^*)$ hold. Thus Theorems 5.7 and 5.8 follow directly from Theorems 5.2 and 5.3. □

6. Bounding the number of lattice points on a curve by curvature and arclength.

Theorem 6.1. *Let $\mathcal{C}$ be a convex curve whose radius of curvature satisfies $\rho \geq R_1$ for some constant $R_1 > 0$ and whose total curvature satisfies $\int_{\mathcal{C}} \kappa \, ds \leq \pi$. Let $\mathcal{L}$ be a lattice. If*

$$\frac{\mathrm{Length}(\mathcal{C})}{(A_{\mathcal{L}} R_1)^{1/3}} \leq 2$$

then $\mathcal{C}$ contains at most two points of $\mathcal{L}$.

Proof. Towards a contradiction assume $\mathcal{C}$ contains three points P_0, P_1, and P_2 of $\mathcal{L}$ and that P_1 is between P_0 and P_2 on $\mathcal{C}$. Let R be the radius of the circle through these points. Because the total curvature of $\mathcal{C}$ is at most π Theorem 4.8 yields a point is a point of $\mathcal{C}$ with radius of curvature R which implies $R \geq R_1$. As the points P_0, P_1, and P_2 are in $\mathcal{L}$ the lower bound $\mathrm{Area}(\triangle P_0 P_1 P_2) \geq A_{\mathcal{L}}/2$ holds by Proposition 3.1 and

$$\frac{A_{\mathcal{L}}}{2} \leq \mathrm{Area}(\triangle P_0 P_1 P_2) < \frac{(\|P_1 - P_0\| + \|P_2 - P_1\|)^3}{16 R_1} \leq \frac{\mathrm{Length}(\mathcal{C})^3}{16 R_1}$$

which contradicts $\mathrm{Length}(\mathcal{C})/(A_{\mathcal{L}} R_1)^{1/3} \leq 2$. □

Theorem 6.2. *Let $\mathcal{C}$ be an open convex curve such that the radius of convergence of $\mathcal{C}$ satisfies the inequality $\rho \geq R_1$ and let $\mathcal{L}$ be a lattice. Then*

$$\#(\mathcal{C} \cap \mathcal{L}) < 4 + \frac{\mathrm{Length}(\mathcal{C})}{(A_{\mathcal{L}} R_1)^{1/3}}. \tag{6.1}$$

If the total curvature of $\mathcal{C}$ satisfies $\boldsymbol{\tau}(\mathcal{C}) \leq \pi$ this can be improved to

$$\#(\mathcal{C} \cap \mathcal{L}) < 2 + \frac{\mathrm{Length}(\mathcal{C})}{(A_{\mathcal{L}} R_1)^{1/3}}. \tag{6.2}$$

Proof. Let $N = \#(\mathcal{C} \cap \mathcal{L})$ and let $P_1, P_2, \ldots, P_N$ be the points of $\mathcal{C} \cap \mathcal{L}$ listed in order along $\mathcal{C}$. Let r_j be the radius of the circle through P_j, P_{j+1}, and P_{j+2}. If the total curvature of $\mathcal{C}\big|_{P_j}^{P_{j+2}}$ is $\leq \pi$ then Theorem 4.8 gives a point Q_j on this curve with $r_j = \rho(Q_j) \geq R_1$. Also, by Proposition 2.3 $\mathrm{Area}(\triangle P_j P_{j+1} P_{j+2}) \geq A_{\mathcal{L}}/2$. Therefore, if the total curvature of $\mathcal{C}\big|_{P_j}^{P_{j+2}}$ is $\leq \pi$ for $j \in \{1, 2, \ldots, N-2\}$ Theorem 3.3 applies and the inequality (6.2) holds. Thus will be the case if $\boldsymbol{\tau}(\mathcal{C}) \leq \pi$.

This leaves the case where for some $k \in \{1, 2, \ldots, N-2\}$ the total curvature of $\mathcal{C}\big|_{P_k}^{P_{k+2}}$ is greater than π. As the total curvature of $\mathcal{C}$ satisfies

$\boldsymbol{\tau}(\mathcal{C}) < 2\pi$ at least one of the arcs $\mathcal{C}\big|_{P_1}^{P_{k+1}}$ or $\mathcal{C}\big|_{P_{k+1}}^{P_N}$ will have total curvature $< \pi$. We prove the case where $\mathcal{C}\big|_{P_1}^{P_{k+1}}$ has total curvature $< \pi$, the other case being similar. Then both $\mathcal{C}\big|_{P_1}^{P_{k+1}}$ and $\mathcal{C}\big|_{P_{k+2}}^{P_N}$ will have total curvature $< \pi$ and by what we have just done

$$\#(\mathcal{C}\big|_{P_1}^{P_{k+1}}) < 2 + \frac{\operatorname{Length}(\mathcal{C}\big|_{P_1}^{P_{k+1}})}{(A_{\mathcal{L}}R_1)^{1/3}}$$
$$\#(\mathcal{C}\big|_{P_{k+2}}^{P_N}) < 2 + \frac{\operatorname{Length}(\mathcal{C}\big|_{P_{k+2}}^{P_N})}{(A_{\mathcal{L}}R_1)^{1/3}}.$$

Adding these and using $\operatorname{Length}(\mathcal{C}\big|_{P_1}^{P_{k+1}}) + \operatorname{Length}(\mathcal{C}\big|_{P_{k+2}}^{P_N}) < \operatorname{Length}(\mathcal{C})$ shows the bound (6.1) holds. □

Theorem 6.3. *Let $\mathcal{C}$ be a closed convex curve whose radius of curvature satisfies $\rho \geq R_1$ for some positive constant R_1 and let $\mathcal{L}$ be a lattice. Then*

$$\#(\mathcal{C} \cap \mathcal{L}) < \frac{\operatorname{Length}(\mathcal{C})}{(A_{\mathcal{L}}R_1)^{1/3}}. \tag{6.3}$$

Proof. Let $P_1, P_2, \dots, P_N$ be the points of $\mathcal{C} \cap \mathcal{L}$ listed in cyclic order around $\mathcal{C}$. By Corollary 4.7 the circle through P_j, P_{j+1}, and P_{j+2} has radius $\rho_{\mathcal{C}}(Q)$ for some point Q on $\mathcal{C}$ and therefore this radius is at least R_1. By Proposition 2.3 the area of $\triangle P_j P_{j+1} P_{j+2}$ is at least $A_{\mathcal{L}}/2$. Therefore Theorem 3.2 implies (6.3). □

Corollary 6.4. *In Theorems 6.1, 6.2, and 6.3 if there is also an upper bound $\rho \leq R_2$ on the radius of curvature, then the theorems still hold if the expression*

$$\frac{L}{(A_{\mathcal{L}}R)^{1/3}}$$

is replaced by either of the expressions

$$\left(\frac{\boldsymbol{\tau}(\mathcal{C})R_2}{R_1}\right)^{1/3} L^{2/3}, \qquad \frac{\boldsymbol{\tau}(\mathcal{C})R_2}{(A_{\mathcal{L}}R_1)^{1/3}}.$$

Proof. This follows from the inequalities of Proposition 4.2. □

Remark 6.5. The expression $\left(\frac{\boldsymbol{\tau}(\mathcal{C})R_2}{R_1}\right)^{1/3} L^{2/3}$ is of interest as the coefficient $\left(\frac{\boldsymbol{\tau}(\mathcal{C})R_2}{R_1}\right)^{1/3}$ is invariant under dilations of the curve. The expression $\frac{\boldsymbol{\tau}(\mathcal{C})R_2}{(A_{\mathcal{L}}R_1)^{1/3}}$ is interesting as it only depends on the integral of curvature $\int_{\mathcal{C}} \kappa\, ds$ and the curvature bounds $1/R_2 \leq \kappa \leq 1/R_1$.

7. Bounding the number of lattice points near a curve.

Lemma 7.1. *Let P_1, P_1, P_3, P_3' be points in $\mathbb{R}^2$. Let $\delta \geq 0$ and $\|P_3 - P_3'\| \leq \delta$. Denote by A the area of $\triangle P_1P_2P_3$, and by A_1 the area of $\triangle P_1P_2P_3'$. Then,*

$$|A - A_1| \leq \frac{\|P_1 - P_2\|\delta}{2}.$$

Proof. Let $\overleftrightarrow{P_1P_2}$ be the line through P_1 and P_2 and let h be the distance of P_3 and h' the distance of P_3' from this line. Then

$$A = \frac{\|P_1 - P_2\|h}{2} \qquad \text{and} \qquad A_1 = \frac{\|P_1 - P_2\|h'}{2}.$$

The distance between P_3 and P_3' is at most δ and therefore $|h - h'| \leq \delta$. From this it follows

$$|A - A_1| = \frac{\|P_1 - P_2\||h - h'|}{2} \leq \frac{\|P_1 - P_2\|\delta}{2}.$$

□

Lemma 7.2. *Let $\triangle P_1P_2P_3$ and $\triangle P_1'P_2'P_3'$ be triangles in the plane with areas A and A' respectively. Let $\delta \geq 0$ and assume*

$$\|P_j - P_j'\| \leq \delta \quad \textit{for} \quad j = 0, 1, 2.$$

Then

$$|A - A'| \leq \frac{(\|P_1 - P_2\| + \|P_3 - P_2\| + \|P_3 - P_1\|)\delta}{2} + \frac{3\delta^2}{2} \tag{7.1}$$

$$\leq (\|P_2 - P_1\| + \|P_3 - P_2\|)\delta + \frac{3\delta^2}{2} \tag{7.2}$$

Proof. Let $A = A_0$ be the area of $\triangle P_1P_2P_3$, A_1 the area of $\triangle P_1P_2P_3'$, A_2 the area of $\triangle P_1P_2'P_3'$, and $A_3 = A'$ the area of $\triangle P_1'P_2'P_3'$. By Lemma 7.1 and the triangle inequality

$$\begin{aligned}
|A_0 - A_1| &\leq \frac{\|P_1 - P_2\|\delta}{2} \\
|A_1 - A_2| &\leq \frac{\|P_1 - P_3'\|\delta}{2} \leq \frac{(\|P_1 - P_3\| + \delta)\delta}{2} \\
|A_2 - A_3| &\leq \frac{\|P_2' - P_3'\|\delta}{2} \leq \frac{(\|P_2 - P_3\| + 2\delta)\delta}{2}
\end{aligned}$$

Therefore

$$\begin{aligned}
|A - A'| &= |A_0 - A_3| \\
&\leq |A_0 - A_1| + |A_1 - A_2| + |A_2 - A_3| \\
&\leq \frac{(\|P_1 - P_2\| + \|P_3 - P_2\| + \|P_3 - P_1\|)\delta}{2} + \frac{3\delta^2}{2}
\end{aligned}$$

which proves the inequality (7.1). By the triangle inequality $\|P_3 - P_1\| \leq \|P_2 - P_1\| + \|P_3 - P_2\|$ and therefore the inequality (7.2) follows from (7.1). □

Lemma 7.3. *In the triangle $\triangle P_1P_2P_2$ as in Figure 11 using the side $\overline{P_1P_3}$ as a base, the height is*

$$h = \frac{a^2}{2R}.$$

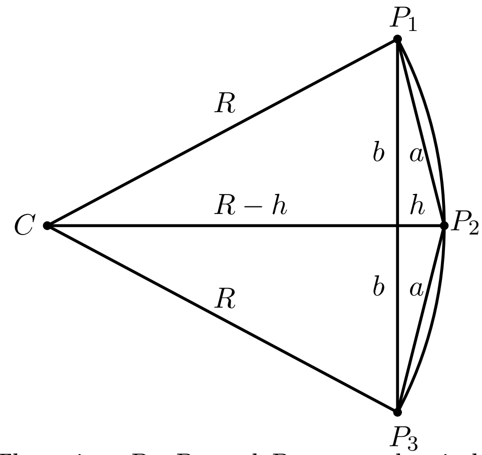

Figure 11. The points P_1, P_2, and P_3 are on the circle of radius R centered at C and $\|P_2 - P_1\| = \|P_3 - P_2\| = a$, $\|P_1 - P_3\| = 2b$, and h is as shown. Then $h = a^2/(2R)$.

Proof. Two applications of the Pythagorean Theorem give

$$b^2 + (R-h)^2 = R^2, \qquad b^2 + h^2 = a^2$$

Solving these for b^2 and setting the results equal gives

$$R^2 - (R-h)^2 = a^2 - h^2$$

and solving this for h gives the desired formula. □

Lemma 7.4. *Let P_1, P_2, P_3 be points on a circle of radius R and let P_1', P_2', P_3' be points with $\|P_j' - P_k'\| \geq d$ when $j \neq k$ for some $d > 0$ and $\|P_j - P_j'\| \leq \delta$ for $j = 1, 2, 3$. Then*

$$\delta < \frac{d^2}{2\big(R + d + \sqrt{(R+d)^2 - d^2}\,\big)} \tag{7.3}$$

implies the points P_1', P_2', and P_3' are not collinear.

Proof. First note

$$\frac{d^2}{2\big(R + d + \sqrt{(R+d)^2 - d^2}\,\big)} < \frac{d^2}{2d} = \frac{d}{2}$$

and thus the inequality (7.3) implies $\delta < d/2$. Whence

$$\|P_j - P_k\| \geq \|P_j' - P_k'\| - \|P_j - P_j'\| - \|P_k - P_k'\| > d - 2\delta > 0.$$

Therefore, the point P_1, P_2, and P_3 are distinct.

Given three points on a circle, then at least one of the points, P, is such that the other two are on opposite sides of the diameter through P. For if for one of the points, call it Q the other two are both on the same side of the diameter through Q, or one of them is the other end of the diameter through Q, then all three are on a closed half circle. Then let P be the one of the points on this half circle which is between the other two.

Therefore, we can label the points P_1, P_2 and P_3 so that P_1 and P_3 are on opposite sides of the diameter through P_2. Farther, we can assume the circle containing P_1, P_2, and P_3 goes through the origin, that P_2 is at the origin, the circle is above the x-axis and is tangent to the x-axis at P_2, P_1 is on the open half plane defined by $x < 0$, and P_3 is on the open half plane defined by $x > 0$. Let $a = d - 2\delta$ and let Q_1 and Q_3 be the points on the circle with $\|P_2 - Q_1\| = \|P_2 - Q_3\| = a$ as in Figure 12. Let h be the distance between P_2 and the line through Q_1 and Q_3.

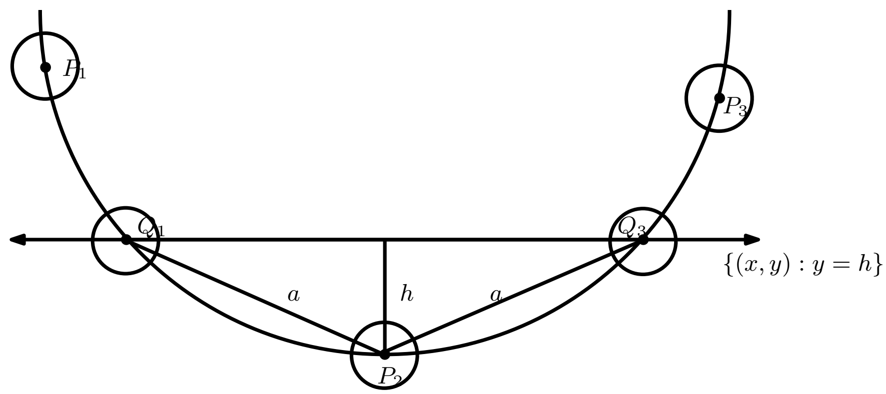

Figure 12. The points Q_1 and Q_3 are the points on the circle such that $\|Q_j - P_2\| = d - 2\delta =: a$. The circles around the points P_j and Q_j have radius δ and therefore contain the points P_1', P_2', and P_3'.

By Lemma 7.3

$$h = \frac{a^2}{2R} = \frac{(d-2\delta)^2}{2R}.$$

As long at $h > 2\delta$ the line with equation $y = h/2$ separates the open disks of radius δ about P_1 and P_3 from the open disk of radius δ about P_2 and therefore the points P_1' and P_3' are above the line $\{y = h/2\}$ and P_2' is below this line and thus these three points are not collinear.

The inequality $h > 2\delta$ is

$$2\delta < \frac{(d-2\delta)^2}{2R} \tag{7.4}$$

which is equivalent to

$$0 < 4\delta^2 - 4(R+d)\delta + d^2.$$

Viewing the right-hand side of this as a quadratic polynomial in δ with roots $r_1 < r_2$, then

$$r_1, r_2 = \frac{R + d \pm \sqrt{(R+d)^2 - d^2}}{2}.$$

Thus, the inequality (7.4) has as solution set the union $(-\infty, r_1) \cup (r_2, \infty)$. But $r_2 > d/2$ so (r_2, ∞) can be ignored. Therefore

$$\delta < r_1 = \frac{R + d - \sqrt{(R+d)^2 - d^2}}{2} = \frac{d^2}{2\big(R + d + \sqrt{(R+d)^2 - d^2}\,\big)}$$

implies the points P'_1, P'_2, and P'_3 are not collinear. □

Lemma 7.5. *Let $\mathcal{C}$ be either a closed convex curve, or a convex arc with total curvature $\leq \pi$ and assume the radius of curvature of $\mathcal{C}$ satisfies*

$$R_1 \leq \rho \leq R_2.$$

Let $\mathcal{L}$ be a lattice and let P'_1, P'_2, and P'_3 be distinct points in $\mathcal{L}$ with

$$\operatorname{dist}(P'_j, \mathcal{C}) < \delta$$

where

$$\delta < \frac{d_{\mathcal{L}}^2}{2\big(R_2 + d_{\mathcal{L}} + \sqrt{(R_2 + d_{\mathcal{L}})^2 - d_{\mathcal{L}}^2}\,\big)}$$

and let P_j be a point of $\mathcal{C}$ with $\|P_j - P'_j\| < \delta$ for $j = 1, 2, 3$. Then,

$$\operatorname{Area}(\triangle P_1 P_2 P_3) \geq \frac{A_{\mathcal{L}}}{2} - (\|P_2 - P_1\| + \|P_3 - P_2\|)\delta - \frac{3}{2}\delta^2.$$

Proof. Let R be the radius of the circle through P_1, P_2, and P_3. Then by Corollary 4.7 or Theorem 4.8 there is a point on $\mathcal{C}$ where $\rho = R$ and thus $R_1 \leq R \leq R_2$. Then by Lemma 7.4 the points P'_1, P'_2, and P'_3 are not collinear and therefore by Proposition 2.3 $\operatorname{Area}(\triangle P'_1 P'_2 P'_3) \geq A_{\mathcal{L}}/2$. The lower bound on the $\operatorname{Area}(\triangle P_1 P_2 P_3)$ follows from Proposition 7.1. □

Theorem 7.6. *Let $\mathcal{C}$ be a convex arc with total curvature $\leq \pi$ and whose radius of curvature satisfies $R_1 \leq \rho \leq R_2$. Let $\mathcal{L}$ be a lattice and let $\delta > 0$ satisfy*

$$\delta < \frac{d_{\mathcal{L}}^2}{2\big(R_2 + d_{\mathcal{L}} + \sqrt{(R_2 + d_{\mathcal{L}})^2 - d_{\mathcal{L}}^2}\,\big)}.$$

Let $L = \operatorname{Length}(\mathcal{C})$. Then

$$\frac{L^3}{8R_1} + 2L\delta + 3\delta^2 \leq A_{\mathcal{L}} \tag{7.5}$$

implies

$$\#\{Q \in \mathcal{L} : \operatorname{dist}(Q, \mathcal{C}) < \delta\} \leq 2. \tag{7.6}$$

Proof. We prove the contrapositive: If $\#\{Q \in \mathcal{L} : \mathrm{dist}(Q, \mathcal{C}) < \delta\} \geq 3$, then the inequality (7.5) is violated. Assume three are three or more points of $\mathcal{L}$ at a distance less than δ form $\mathcal{C}$. Then there are P_1, P_2, P_3, P_1', P_2', and P_3' that satisfy the hypothesis of Lemma 7.5. Let R be the radius of the circle through P_1, P_2, and P_3. By Theorem 4.8 and the given bounds on ρ we have $R_1 \leq R \leq R_2$. By Lemma 7.5 and Proposition 3.1

$$\begin{aligned}\frac{A_{\mathcal{L}}}{2} - L\delta - \frac{3}{2}\delta^2 &\leq \frac{A_{\mathcal{L}}}{2} - (\|P_2 - P_1\| + \|P_3 - P_2\|)\delta - \frac{3}{2}\delta^2 \\ &\leq \mathrm{Area}(\triangle P_1P_2P_3) \\ &< \frac{(\|P_2 - P_1\| + \|P_3 - P_2\|)^3}{16R} \\ &\leq \frac{L^3}{16R_1}\end{aligned}$$

which contradicts (7.5). □

Theorem 7.7. *Let $\mathcal{C}$ be a convex arc with total curvature at most π with radius of curvature bounded by $R_1 \leq \rho \leq R_2$. Let $\mathcal{L}$ be a lattice and $\delta > 0$ with*

$$\delta < \frac{d_{\mathcal{L}}^2}{2(R_2 + d_{\mathcal{L}} + \sqrt{(R_2 + d_{\mathcal{L}})^2 - d_{\mathcal{L}}^2})}$$

and

$$\frac{A_{\mathcal{L}}}{2} - L\delta - \frac{3}{2}\delta^2 > 0. \tag{7.7}$$

Then,

$$\#\{Q \in \mathcal{L} : \mathrm{dist}(\mathcal{C}, Q) < \delta\} < 2 + \frac{L}{\big(R_1(A_{\mathcal{L}} - 2L\delta - 3\delta^2)\big)^{1/3}}$$

where $L = \mathrm{Length}(\mathcal{C})$.

Proof. Let $N = \#\{Q \in \mathcal{L} : \mathrm{dist}(\mathcal{C}, Q) < \delta\}$ and

$$\{Q \in \mathcal{L} : \mathrm{dist}(\mathcal{C}, Q) < \delta\} = \{P_1', P_2', \dots, P_N'\}$$

and let P_j be a point of $\mathcal{C}$ with $\mathrm{dist}(P_j', \mathcal{C}) = \|P_j - P_j'\|$. By Lemma 7.5

$$\mathrm{Area}(\triangle P_jP_{j+1}P_{j+2}) \geq \frac{A_{\mathcal{L}}}{2} - (\|P_{j+1} - P_j\| + \|P_{j+2} - P_{j+1}\|)\delta - \frac{3}{2}\delta^2.$$

and by Theorem 4.8 the circle through P_j, P_{j+1} and P_{j+2} has radius of curvature $\rho(P)$ for some point on $\mathcal{C}$ and therefore its radius is $\geq R_1$. Thus, the result follows from Theorem 3.3. □

Theorem 7.8. *Let $\mathcal{C}$ be a closed convex curve with radius of curvature bounded by $R_1 \leq \rho \leq R_2$. Let $\mathcal{L}$ be a lattice and $\delta > 0$ with*

$$\delta < \frac{d_{\mathcal{L}}^2}{2(R_2 + d_{\mathcal{L}} + \sqrt{(R_2 + d_{\mathcal{L}})^2 - d_{\mathcal{L}}^2})}$$

and

$$\frac{A_{\mathcal{L}}}{2} - L\delta - \frac{3}{2}\delta^2 > 0. \tag{7.8}$$

Then,

$$\#\{Q \in \mathcal{L} : \operatorname{dist}(\mathcal{C}, Q) < \delta\} < \frac{L}{\big(R_1(A_{\mathcal{L}} - 2L\delta - 3\delta^2)\big)^{1/3}}$$

where $L = \text{Length}(\mathcal{C})$.

Proof. Other than using Theorem 3.2 rather than Theorem 3.3 this is exactly the same as Theorem 7.7. □

Using the inequalities of Proposition 4.2, in particular $L \le \boldsymbol{\tau}(\mathcal{C})R_2$, we obtain corollaries to the previous three theorems that only involve the total curvature, $\boldsymbol{\tau}(\mathcal{C})$, the curvature bounds R_1 and R_2 and the invariants $A_{\mathcal{L}}$ and $d_{\mathcal{L}}$ of the lattice.

Corollary 7.9. *In Theorem 7.6 if the inequality* (7.5) *is replaced by*

$$\frac{(\boldsymbol{\tau}(\mathcal{C})R_2)^3}{8R_1} + 2\,\boldsymbol{\tau}(\mathcal{C})R_2\delta + 3\delta^2 \le A_{\mathcal{L}}$$

the conclusion $\#\{Q \in \mathcal{L} : \operatorname{dist}(Q, \mathcal{C}) < \delta\} \le 2$ *still holds.* □

Corollary 7.10. *In Theorem 7.7 if the hypothesis 7.7 is replaced by*

$$\frac{A_{\mathcal{L}}}{2} - 2\,\boldsymbol{\tau}(\mathcal{C})R_2\delta - \frac{3}{2}\delta^2 > 0$$

then

$$\#\{Q \in \mathcal{L} : \operatorname{dist}(\mathcal{C}, Q) < \delta\} < 2 + \frac{R_2\,\boldsymbol{\tau}(\mathcal{C})}{(R_1(A_{\mathcal{L}} - 2\,\boldsymbol{\tau}(\mathcal{C})R_2\delta - 3\delta^2))^{1/3}}.$$

□

Corollary 7.11. *In Theorem 7.8 if the hypothesis 7.8 is replaced by*

$$\frac{A_{\mathcal{L}}}{2} - 2\,\boldsymbol{\tau}(\mathcal{C})R_2\delta - \delta^2 > 0$$

then

$$\#\{q \in \mathcal{L} : \operatorname{dist}(\mathcal{C}, Q) < \delta\} < \frac{R_2\,\boldsymbol{\tau}(\mathcal{C})}{(R_1(A_{\mathcal{L}} - 2\,\boldsymbol{\tau}(\mathcal{C})R_2\delta - 3\delta^2))^{1/3}}.$$

□

We record what is the specialization of these results to circles.

Corollary 7.12. *Assume* $\mathcal{C}$ *is an arc of a circle of radius* R, $\mathcal{L}$ *is a lattice, and* $\delta > 0$ *satisfies*

$$\delta < \frac{d_{\mathcal{L}}^2}{2(R + d_{\mathcal{L}} + \sqrt{(R + d_{\mathcal{L}})^2 - d_{\mathcal{L}}^2})}.$$

Let $L = \text{Length}(\mathcal{C})$. *Then,*

(a) *If $L \leq \pi R$ (which is equivalent to having total curvature $\leq \pi$) and*

$$\frac{L^3}{8R} + 2L\delta + 3\delta^2 \leq A_{\mathcal{L}}$$

then

$$\#\{Q \in \mathcal{L} : \operatorname{dist}(Q, \mathcal{C}) < \delta\} \leq 2.$$

(b) *If $L \leq \pi R$ and*

$$\frac{A_{\mathcal{L}}}{2} - 2L\delta - 3\delta^2 > 0,$$

then

$$\#\{Q \in \mathcal{L} : \operatorname{dist}(\mathcal{C}, Q) < \delta\} < 2 + \frac{L}{(R(A_{\mathcal{L}} - 2L\delta - 3\delta^2))^{\frac{1}{3}}}.$$

(c) *If $\mathcal{C}$ is the entire circle (i.e. $L = 2\pi R$) and*

$$\frac{A_{\mathcal{L}}}{2} - 4\pi R\delta - 3\delta^2 > 0,$$

then

$$\#\{Q \in \mathcal{L} : \operatorname{dist}(\mathcal{C}, Q) < \delta\} < \frac{2\pi R^{\frac{2}{3}}}{(A_{\mathcal{L}} - 4\pi R - 3\delta^2)^{\frac{1}{3}}}.$$

8. Examples

The following shows that Theorem 6.2 and Corollary 6.4 are close to being sharp.

Theorem 8.1. *Let $\mathcal{L}$ be a lattice and $n \geq 2$ an integer. Then there is a convex curve $\mathcal{C}$ of length L that contains exactly n points of $\mathcal{L}$, and has lower and upper bounds*

$$R_1 = \min_{P \in \mathcal{C}} \rho(P), \qquad R_2 = \max_{P \in \mathcal{C}} \rho(P)$$

for the radius of curvature of $\mathcal{C}$, so that the inequalities

$$\frac{L}{(A_{\mathcal{L}} R_1)^{1/2}} \leq \left(\frac{\boldsymbol{\tau}(\mathcal{C}) R_2}{(A_{\mathcal{L}} R_1)}\right)^{1/3} L^{2/3} \leq \frac{\boldsymbol{\tau}(\mathcal{C}) R_2}{(A_{\mathcal{L}} R_1)^{1/3}} < n + 2.$$

hold.

Proof. In light of Proposition 4.2 we only need to find an example with

$$\frac{\boldsymbol{\tau}(\mathcal{C}) R_2}{(A_{\mathcal{L}} R_1)^{1/3}} < n + 2 \tag{8.1}$$

Let $\mathcal{L} = \mathcal{L}(v_0, v_1, v_2)$ where we can assume $v_1 \wedge v_2 > 0$, by possibly replacing v_2 by $-v_2$. Then $v_1 \wedge v_2 = A_{\mathcal{L}}$. Let $a > 0$ and $b = a + (n - 1)$. Define a curve $\mathcal{C}_a$ parametrically $c \colon [a, b] \to \mathbb{R}^2$ by

$$c_a(t) = v_0 + tv_1 + \frac{t(t+1)}{2} v_2.$$

Each of the points $c_a(k)$ with $k = a, a+1, \ldots, a+(n-1)$ is a point of $\mathcal{L}$ and if $c_a(t) = v_0 + tv_1 + (t(t+1)/2)v_2 v_0 = v_0 + kv_1 + mv_2$ is a point of $\mathcal{L}$ on $\mathcal{C}_a$, then the linear independence of v_1 and v_2 implies $k = t$ and $m = t(t+1)/2$, so that $c_a(t) = c_a(k)$. Thus, there are exactly n points of $\mathcal{L}$ on $\mathcal{C}_a$. We will show if a is sufficiently large this curve has the desired properties. The derivatives of c_a are

$$c_a'(t) = v_1 + (t+1/2)v_2, \qquad c_a''(t) = v_2$$

Then

$$\lim_{a\to\infty} \frac{\|c_a'(t)\|}{a} = \lim_{a\to\infty} \left\| \frac{1}{a}v_1 + \frac{t+1/2}{a}v_2 \right\| = \|v_2\|$$

and this limit holds uniformly in $t \in [a,b]$. This gives the asymptotic formula

$$\|c_a'(t)\| \sim a\|v_2\|$$

and this holds uniformly for $t \in [a,b]$. Using a standard formula for curvature

$$\rho = \frac{1}{\kappa} = \frac{\|c_a'(t)\|^3}{c_a'(t) \wedge c_a''(t)} \sim \frac{a^3\|v_2\|^3}{A_{\mathcal{L}}}$$

and this holds uniformly in $t \in [a,b]$. As this formula is independent of t we see that if $R_1(a)$ and $R_2(a)$ are the minimum and maximum radius of curvature on $\mathcal{C}_a$ then

$$R_1(a) \sim R_2(a) \sim \frac{a^3\|v_2\|^3}{A_{\mathcal{L}}}.$$

Asymptotically the total curvature of $\mathcal{C}_a$ is

$$\begin{aligned}\tau(a) = \int_{\mathcal{C}_a} \frac{ds}{\rho} &= \int_a^b \frac{\|c_a'(t)\|\,dt}{\left(\dfrac{\|c_a'(t)\|^3}{A_{\mathcal{L}}}\right)} = A_{\mathcal{L}} \int_a^b \frac{dt}{\|c_a'(t)\|^2} \\ &\sim A_{\mathcal{L}} \int_a^b \frac{dt}{a^2\|v_2\|^2} = \frac{(n+1)A_{\mathcal{L}}}{a^2\|v_2\|^2}\end{aligned}$$

where we have used $b - a = n+1$. Putting these formulas together gives

$$\frac{\tau(a)R_2(a)}{(A_{\mathcal{L}}R_1(a))^{1/3}} \sim \frac{\left(\dfrac{(n+1)A_{\mathcal{L}}}{a^2\|v_2\|^2}\right)\left(\dfrac{a^3\|v_2\|^3}{A_{\mathcal{L}}}\right)}{\left(A_{\mathcal{L}}\dfrac{a^3\|v_2\|^3}{A_{\mathcal{L}}}\right)^{1/3}} = n+1.$$

Thus for sufficiently large a the inequality (8.1) holds. □

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