

# A Reverse Isoperimetric Inequality, Stability and Extremal Theorems for Plane Curves with Bounded Curvature

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In this note we discuss some elementary theorems about the relation between area and length of closed embedded plane curves with bounded curvature. Our main result (see Theorem 4.1) solves the extremal problem of which domain has *largest* boundary length among embedded disks in the plane whose boundary curvatures are uniformly bounded and whose area is fixed and sufficiently small.

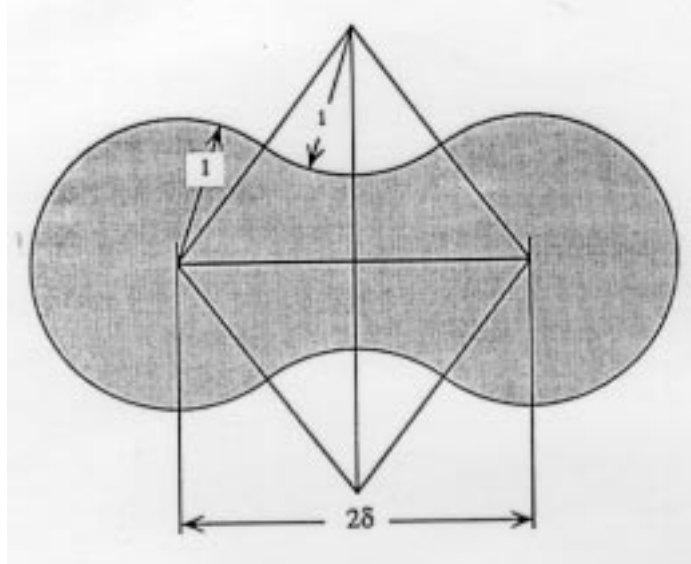


Fig. 1 “Peanut” domain  $P_\delta$ .

**Reverse Isoperimetric Inequality.** *If  $M$  is an embedded closed disk in the plane  $\mathbf{R}^2$  whose boundary curvature satisfies  $|\kappa| \leq 1$  and with area  $A \leq \pi + 2\sqrt{3}$  then the length of  $\partial M$  is bounded by*

$$\frac{L - 2\pi}{4} \leq \text{Arcsin} \left( \frac{A - \pi}{4} \right).$$

*If equality holds then  $M$  is congruent to the peanut shaped domain of Figure 1.*

This gives an estimate in the reverse direction to the classical isoperimetric inequality. There is also a threshold phenomenon: if the area is larger than  $\pi + 2\sqrt{3}$  then there is no upper bound for the length of  $\partial M$ . This is the area of the pinched peanut domain  $P_{\sqrt{3}}$ . Examples can be found by breaking a thin peanut and connecting the ends with a long narrow strip. In fact, the set of possible points  $(A, L)$  for embedded disks whose boundary satisfies  $|\kappa| \leq 1$  is further restricted (Theorem 4.1). There is a suggestive physical interpretation of the equivalent dual problem, where the length is fixed and the minimal area disk is sought. One may imagine the cross section of a hose in which the inside pressure is smaller than the outside. If the hose has limited flexibility, modelled by a uniform bound on the curvature, then the equilibrium section is again the peanut shape.

In Section 4. we prove existence and uniqueness of the extremal figures. We use a replacement argument to show that extremals are piecewise circular arcs. Compactness depends on a priori length bounds. In Section 3. we consider length estimates and some related stability results in the class of embedded disks whose boundary curvature is uniformly bounded. Our results say that if area or some other quantity is small, such as the circumradius, then the curve must be near the circle. This gives a preliminary reverse isoperimetric inequality which improves with the addition of extra information, say on the circumradius, for this class of curves. The results depend on a theorem of Pestov and Ionin [PI] on the existence of a large disk in a domain with uniformly bounded curvature (see e.g. [BZ].) In Section 2. we include an argument for Pestov and Ionin's theorem along the lines of Lagunov's [L] proof of the higher dimensional generalization using analysis of the structure of the cut set of such a domain. Lagunov gives a sharp lower bound for the radius of the biggest ball enclosed within hypersurfaces all of whose principal curvatures are bounded  $|\kappa_i| \leq 1$ . Lagunov and Fet [LF] show that the bound is increased if additional topological hypotheses are imposed. It is noteworthy that the examples which show the sharpness of the Lagunov and Lagunov-Fet bounds for dimension greater than one are not unit spheres. We indicate how the argument carries over to general Riemannian surfaces. In higher dimensions, Alexander and Bishop [AB] have found inradius bounds depending on the curvatures and topology of the manifold and its boundary. Our results use both the existence of a disk and structure of the cut set. In Section 1. we consider curves which are only continuously differentiable and whose curvature is bounded in an appropriate weak sense which is suitable to extremal problems. Some other extremal problems for such curves have been studied previously. For example, the problem of finding the shortest plane curve with given endpoint and starting line element (position and direction) considered by Markov [P]. The shortest plane curve given starting and ending line elements was found by Dubins [D].

*Acknowledgement.* This study evolved from a question of Paul Fife. Also we thank Stephanie Alexander, Richard Bishop, Nick Korevaar and Tomasz Serbinowski for helpful comments. Typeset by AMS TeX.

### 1. Curves with weakly bounded curvature

Let  $\Sigma$  be 2-manifold of class  $C^2$ . We will usually assume that our curves  $c : (a, b) \rightarrow \Sigma$  are  $C^1$ , parameterized by arclength, such that the tangent vector  $\mathbf{t} = c'$  is absolutely continuous. Let  $\mathbf{n}$  be the unit normal, chosen so that  $\{\mathbf{t}, \mathbf{n}\}$  is a right handed system. Then the defining equation for geodesic curvature  $\nabla_{\mathbf{t}} \mathbf{t}(s) = \kappa_g(s) \mathbf{n}(s)$  implies that  $\kappa_g$  exists almost everywhere as an  $L^1_{\text{loc}}$  function which we shall assume satisfies the  $L^\infty$  bound

$$(1.1) \quad \|\kappa_g\|_\infty \leq 1.$$

Let  $\mathcal{K}$  denote the class of  $C^1$  curves of  $\Sigma$ , parameterized by arclength, whose  $c'$  is absolutely continuous, and whose geodesic curvature satisfies (1.1).

If  $\Sigma = \mathbf{R}^2$  then  $c \in \mathcal{K}$  is equivalent to the condition  $c \in C^1$  and

$$(1.2) \quad |c'(s) - c'(t)| \leq |s - t| \quad \text{for all } s, t$$

which is called a *constraint on average curvature* by Dubins [D]. Since (1.2) implies that the  $c'$  is Lipschitz and thus absolutely continuous, by a theorem of Lebesgue  $c'$  is differentiable almost everywhere, and is the integral of its derivative in the sense that

$$c'(b) - c'(a) = \int_a^b c''(s) ds.$$

By looking at difference quotients and using (1.2) we see that  $|c''(s)| \leq 1$  at all points where it exists. As  $\langle c', c' \rangle \equiv 1$  we also have that  $c'' = \mathbf{t}' \perp \mathbf{t} = c'$  at all points where  $c'' = \mathbf{t}'$  exists. Thus  $\kappa$ , given by  $c'' = \mathbf{t}' = \kappa \mathbf{n}$  is defined almost everywhere. As  $|c''| \leq 1$  this implies  $|\kappa| \leq 1$  at all points where  $c''$  exists, hence (1.1) holds.

Plane curves  $c \in \mathcal{K}$  have a well defined direction angle from which the position can be recovered. Because the rotation  $J$  by  $90^\circ$  is linear,  $\mathbf{n} = J\mathbf{t}$  is differentiable at exactly the same points that  $\mathbf{t}$  is, and  $\mathbf{n}' = J\mathbf{t}' = J\kappa\mathbf{n} = -\kappa\mathbf{t}$ . Thus the usual  $c' = \mathbf{t}$  and  $\mathbf{t}' = \kappa\mathbf{n}$  hold at all points where  $c''$  exists, and thus almost everywhere. Now define a function  $\vartheta(s)$  by

$$\vartheta(s) = \int_0^s \kappa(t) dt$$

where the integral is in the sense of Lebesgue. By another theorem of Lebesgue the function  $\vartheta$  is absolutely continuous and has derivative  $\kappa$  almost everywhere. This implies  $|d\vartheta/ds| = |\kappa| \leq 1$  almost everywhere. Define a functions  $\mathbf{R} \rightarrow \mathbf{R}^2$  by

$$\begin{aligned} \mathbf{t}^*(s) &= (\cos \vartheta(s), \sin \vartheta(s)), \\ \mathbf{n}^*(s) &= (-\sin \vartheta(s), \cos \vartheta(s)). \end{aligned}$$

A unit speed curve  $c^*$  is defined by

$$c^*(s) = \int_0^s \mathbf{t}^*(t) dt.$$

Then  $c^*$ ,  $\mathbf{t}^*$ , and  $\mathbf{n}^*$  satisfy

$$(1.3) \quad c^{*'} = \mathbf{t}^*, \quad \mathbf{t}^{*'} = \kappa \mathbf{n}^*, \quad \mathbf{n}^{*'} = -\kappa \mathbf{t}^*$$

almost everywhere. We can rotate and translate  $c^*$  so that  $c(0) = c^*(0)$  and  $\mathbf{t}(0) = \mathbf{t}^*(0)$ .

Consider the function  $f(s) = |\mathbf{t}(s) - \mathbf{t}^*(s)|^2 + |\mathbf{n}(s) - \mathbf{n}^*(s)|^2$ . Then  $f$  is absolutely continuous and (1.1), (1.3) and a calculation imply that  $f'(s) = 0$  almost everywhere. As  $f$  is absolutely continuous this implies  $f$  is constant. But  $f(0) = 0$  so  $f \equiv 0$ . Therefore  $c' = c^{*'}$  which in turn implies  $c = c^*$ . So  $c'(s) = (\cos \vartheta(s), \sin \vartheta(s))$  where  $d\vartheta/ds = \kappa$  almost everywhere, and

$$(1.4) \quad \vartheta(s) - \vartheta(0) = \int_0^s \kappa(t) dt.$$

Under this hypothesis, Proposition 1.1 and Proposition 1.2, which generalize the classical lemmata of Schur-Schmidt, continue to hold. This is the desirable hypothesis since it is the expected regularity for solutions of optimal control problem with curvature as the control  $\kappa \in [-1, 1]$  for maximal length with the constraints that the area be fixed and the curve be embedded. Part (3) is a special case of a theorem due to A. Schur and E. Schmidt [BL], [C], [d], [Gg] which says the distance between endpoints of a convex planar curve is smaller than the distance between endpoints of a second curve with smaller curvatures at corresponding points.

**Proposition 1.1.** *Let  $\gamma : [0, L] \rightarrow \mathbf{R}^2$  be a curve in  $\mathcal{K}$ . Let  $\gamma(0) = 0$  and  $\gamma'(0) = \partial/\partial x$ . Denote the coordinates of  $\gamma(s) = (x(s), y(s))$ . Then*

- (1)  $x(s) \geq \sin s$  for all  $0 \leq s \leq \pi$ . Equality holds if and only if  $\gamma$  is an arc of a unit circle.
- (2)  $|y(s)| \leq 1 - \cos s$  for  $0 \leq s \leq \pi/2$  with equality if and only if  $\gamma$  is an arc of a unit circle.
- (3)  $|\gamma(s)| \geq 2 \sin(s/2)$  for  $0 \leq s \leq 2\pi$  with equality if and only if  $\gamma$  is an arc of the unit circle.
- (4) In particular, if  $\gamma : [0, \pi] \rightarrow \mathbf{R}^2$  is tangent to a unit circle at  $\gamma(0)$  then  $\gamma(s)$  is disjoint from the interior of the circle for  $0 \leq s \leq \pi$ .

*Proof.* This is a standard fact from elementary differential geometry for  $C^2$  curves and given by Dubins [D] for curves in  $\mathcal{K}$ . We give the proof for completeness sake. By the assumption on curvature and the representation (1.3), the direction of the

tangent vector of the curve  $\vartheta(s) = \int_0^s \kappa(s) ds$  is less than the corresponding angle of a circular arc with unit curvature  $|\vartheta(s)| \leq s$ . Hence

$$x(s) = \int_0^s \cos(\vartheta(s)) ds \geq \int_0^s \cos s ds = \sin s$$

for  $0 \leq s \leq \pi$  with equality if and only if  $|\vartheta| = s$  *a.e.* Similarly

$$|y(s)| = \left| \int_0^s \sin(\vartheta(s)) ds \right| \leq \int_0^s \sin s ds = 1 - \cos s$$

for  $0 \leq s \leq \pi/2$  with equality if and only if  $|\vartheta| = s$  *a.e.* This implies (1) and (2). Now orient the curve so that  $\gamma(s/2) = 0$  and  $\gamma'(s/2) = \partial/\partial x$ . By (1)  $x(s) \geq \sin(s/2)$  and  $x(0) \leq -\sin(s/2)$  which implies (3). Equality implies  $\gamma$  is the arc of a unit circle.  $\square$

This implies a related result which is occasionally useful.

**Proposition 1.2.** *Let  $\gamma : [0, L] \rightarrow \mathbf{R}^2$  be of class  $\mathcal{K}$  ( $\mathbf{t} = \gamma'$  is absolutely continuous, parameterized by arclength and  $|\nabla_{\mathbf{t}} \mathbf{t}| = |\kappa| \leq 1$  *a.e.*) Suppose the curve has endpoints on the boundary of a disk  $B_R$  of radius  $R \leq 1$ , lies outside  $B_R$  and has length*

$$L \leq 2\pi - 2 \sin^{-1} R.$$

Then  $\gamma$  is an arc of a unit circle and either

- (1)  $L = 2\pi - 2 \sin^{-1} R$ ; or
- (2)  $R = 1$ .

*Proof.* By the Schur-Schmidt Proposition 1.1, the endpoints of  $\gamma$  are a distance  $D = \text{dist}(\gamma(0), \gamma(L)) \geq 2 \sin(L/2)$  apart. If  $L > 2 \sin^{-1} R$  then  $D > 2R$ , the diameter of the circle unless (1) holds. On the other hand, a curve of length  $L \leq 2 \sin^{-1} R \leq \pi R$  whose endpoints are on a circle of radius  $R$  can be at most the distance of the chord along the circle apart, namely  $\text{dist}(\gamma(0), \gamma(L)) \leq 2R \sin(L/2R)$  which is a contradiction unless  $R = 1$  and  $\gamma$  is the arc of a unit circle.  $\square$

The strong maximum principle holds for curves  $\gamma \in \mathcal{K}$ . Although this follows from the maximum principle for weak solutions of an elliptic equation, in the curve case it also follows immediately from Proposition 1.1. Suppose  $\gamma \in \mathcal{K}$  so that  $\gamma(0) \in \partial B_1(O)$  and  $\gamma(-\varepsilon, \varepsilon) \subset \overline{B_1}(O)$  for some  $0 < \varepsilon < \pi$ . Then  $\gamma((-\varepsilon, \varepsilon)) \subset \partial B_1(O)$  because  $\gamma$  is tangent to the disk at  $\gamma(0)$ .

## 2. The theorem of Pestov and Ionin and the structure of the cut locus

Let  $M$  be a simply connected plane domain with  $C^1$  boundary which satisfies a one-sided condition on the curvature. Let the boundary curve of  $M$  be positively oriented, parameterized by arclength,  $\gamma'$  absolutely continuous and  $\langle \gamma'(s+h) - \gamma'(s), \mathbf{n}(s) \rangle \leq h$  for all  $s$  and  $0 < h < \pi$ . Equivalently, the boundary  $\partial M$  has curvature satisfying  $\kappa_g \leq 1$  *a.e.* We denote the class of all such curves by  $\mathcal{K}^+$ .

**Proposition 2.1.** (*Pestov and Ionin [PI]*) *Let  $M \subset \mathbf{R}^2$  be an embedded disk whose boundary is of class  $\mathcal{K}^+$ . Then  $M$  contains a disk of radius one. In particular the area of  $M$  is at least  $\pi$  with equality if and only if  $M$  is a disk of radius one.*

*Outline of the proof.* For  $X \in \partial M$  let  $C(X)$  be the first point  $P$  along the inward normal to  $\partial M$  at  $X$  where the segment  $[X, P(X)]$  stops minimizing  $\text{dist}(P, \partial M)$ . Call this the cut point of  $X \in \partial M$  in  $M$ . From the definition it is clear that  $M$  contains a disk of radius  $d(X, C(X))$  about  $C(X)$ . Lemma 2.2 shows that if  $C(X)$  is the cut point of  $X \in \partial M$ , then at least one of the following two conditions holds

- (1)  $C(X)$  is a focal point of  $\partial M$  along the normal line to  $\partial M$  at  $X$ , or
- (2) there is at least one other point  $Y \in \partial M$  so that  $C(Y) = C(X)$  and

$$|C(X) - X| = |C(X) - Y| = \text{dist}(C(X), \partial M).$$

(For example, if the boundary were  $C^2$ , see [CE, Lemma 4.2 page 93].) If  $C(X)$  is a focal point of  $\partial M$  then the curvature condition implies  $|X - C(X)| \geq 1$  by Lemma 2.3 and we are done. However, if  $\mathcal{C}$  denotes the set of all cut points then we will show that  $\mathcal{C}$  contains at least one focal point in Lemma 2.7.  $\square$

We spell out these notions for  $M \subset \mathbf{R}^2$  with boundaries of class  $\mathcal{K}^+$ . In fact, the results of this section apply almost directly for  $M$  which is a  $C^2$  compact two dimensional Riemannian manifold with  $C^1$  nonempty boundary  $\partial M$ . For any  $X \in \partial M$  let  $\eta_X(s)$  be the unit speed geodesic through  $X$  with  $\eta_X'(0)$  equal to the inward unit normal to  $\partial M$ . The *cut point* of  $X \in \partial M$  is the point  $\eta_X(s_0)$  where  $s_0$  is the supremum of all  $s > 0$  so that the segment  $[X, \eta_X(s)]$  realizes the distance  $\text{dist}(\eta_X(s), \partial M)$ . The *focal point* of  $X \in \partial M$  is the point  $\eta_X(s_1)$  where  $s_1$  is supremum of values  $s > 0$  so that the function on  $\partial M$  defined by  $Y \mapsto \text{dist}(\eta_X(s), Y)$  has a local minimum at  $Y = X$ . If  $\partial M$  is  $C^2$  at  $X$  then  $s_1$  is the first  $s$  where  $Y \mapsto \text{dist}(\eta_X(s), Y)$  ceases to have a positive second derivative at  $Y = X$ . It is possible that no such  $s_1$  exists; in this case we say that the *focal distance* is  $s_1 = \infty$ . Clearly  $s_0 \leq s_1$ .

Denote by  $\mathcal{C}$  the set of all cut points of  $\partial M$  in  $M$ . Our goal is to understand what the local geometry of  $\mathcal{C}$  is like at its “nice” points.

**Lemma 2.2.** *Any point  $P \in \mathcal{C}$  satisfies at least one of the following two conditions*

- (1)  $P$  is a focal point of  $\partial M$  or
- (2) There are two or more distance minimizing geodesics from  $\partial M$  to  $P$ .

*Proof.* This is standard. If  $P \in \mathcal{C}$  is not a focal point of  $\partial M$  then let  $r := \text{dist}(P, \partial M)$  and let  $X \in \partial M$  be a point with  $P = \eta_X(r)$ . Then choose a sequence  $s_k \searrow r$  such that for each  $k$  there is a point  $X_k \in \partial M$  so that  $\eta_X(s_k) = \eta_{X_k}(r_k)$  for some  $r_k < s_k$ . By going to a subsequence we can assume that  $X_k \rightarrow Y$  for some  $Y \in \partial M$ . Because  $P$  is not focal point of  $\partial M$  we have  $Y \neq X$ . It follows that  $\eta_Y(r) = P$  and  $\eta_Y$  is a minimizing geodesic from  $\partial M$  to  $P$ .  $\square$

**Lemma 2.3.** *Let  $M \subset \mathbf{R}^2$  be a domain whose boundary is of class  $\mathcal{K}^+$ . Let  $Y \in \mathcal{C}$  be a focal point. Then  $\text{dist}(Y, \partial M) \geq 1$ .*

*Proof.* Let  $Y = \eta_X(s_0)$  for some point  $X \in \partial M$  and  $s_0 > 0$ . Let  $\gamma \in \mathcal{K}^+$  denote the boundary curve  $\partial M$  parameterized so that  $\gamma(0) = X$ . Since  $\gamma$  is tangent to  $\partial M$  at  $X$ , by Proposition 1.1, some interval  $\gamma((-\varepsilon, \varepsilon))$  is not contained in the open disk  $B_s(\eta_X(s))$  for each  $0 < s < 1$ . Hence  $\partial M \ni Z \mapsto \text{dist}(Z, \eta_X(s))$  has a local minimum at  $Z = X$ . Thus  $s_0 \geq 1$ .  $\square$

**Lemma 2.4 (Structure of the cut locus away from focal points.).** *Let  $P \in \mathcal{C}$  be a cut point that is not a focal point and let  $r = \text{dist}(P, \partial M)$ . Then there is a finite number of  $k \geq 2$  of minimizing geodesics from  $P$  to  $\partial M$ , and*

Case 1: *If  $k = 2$ , then there is a neighborhood  $U$  of  $P$  so that  $\mathcal{C} \cap U$  is a  $C^1$  curve and the tangent to  $\mathcal{C}$  at  $P$  bisects the angle between the two minimizing geodesics from  $P$  to  $\partial M$ .*

Case 2: *If  $k \geq 3$ , then the  $k$  geodesic segments from  $P$  to  $\partial M$  split the disk  $B_r(P)$  into  $k$  sectors  $S_1, \dots, S_k$ . There is a small open disk  $U$  about  $P$  so that in each sector  $S_i$  the set  $\mathcal{C} \cap U \cap S_i$  is a  $C^1$  curve ending at  $P$  and the tangent to this curve at  $P$  is the angle bisector of the two sides of the sector  $S_i$  at  $P$ .*

*Remark 2.5.* When viewed correctly, this is not a surprising result. In the Euclidean case take  $k$  points  $X_1, \dots, X_k$  on the boundary of a disk  $B_r(P)$  that divides the disk into sectors  $S_1, \dots, S_k$ . The cut set  $\mathcal{C}$  of the finite set  $\{X_1, \dots, X_k\}$  is exactly the union of the angle bisectors of the sectors  $S_i$ . The theorem just says that away from focal points this model is correct at the infinitesimal level.

*Proof.* By Lemma 2.2 there are at least two minimizing geodesics from  $P$  to  $\partial M$ . If there were an infinite number of these segments, then their endpoints would accumulate at some point  $Y \in \partial M$ . Then  $P$  would be a focal point of  $Y$ . Thus the number of such segments is finite.

Let  $X_1, \dots, X_k$  be the points in  $\partial B_r(P) \cap \partial M$  (so that  $d(P, X_i) = \text{dist}(P, \partial M)$ ). Let  $\varepsilon > 0$  be small. There is then an  $r_1 > r$  so that

$$(2.1) \quad B_{r_1}(P) \cap \partial M \subseteq \bigcup_{k=1}^k B_\varepsilon(X_i).$$

Therefore if  $\delta = \frac{1}{2}(r_1 - r)$  and  $Q \in B_\delta(P)$  then the point of  $\partial M$  closest to  $Q$  is in  $\partial M \cap \bigcup_1^k B_\varepsilon(X_i)$ . For let  $X \in \partial M$  be the point closest to  $Q$ , then

$$d(Q, X) \leq d(Q, X_1) \leq d(Q, P) + d(P, X_1) \leq \frac{\delta}{2} + r < r_1.$$

Thus (1.1) implies  $X \in \partial M \cap \bigcup_1^k B_\varepsilon(X_i)$ . In what follows we will take a point  $Q$  close to  $P$  and assume that the point of  $\partial M$  closest to  $Q$  is close to one of the points  $X_1, \dots, X_k$ . This is justified in light of the remarks just made.

If  $k = 2$  then  $P = \eta_{X_1}(r) = \eta_{X_2}(r)$  for  $X_1, X_2 \in \partial M$ . Let  $c_i$  be a small piece of  $\partial M$  containing  $X_i$  and let  $\rho_i(Q)$  be the distance of  $Q \in M$  from  $c_i$ . Then, as  $P$  is not a focal point, the function  $\rho_i$  is  $C^1$  in a neighborhood of the minimizing geodesic from  $X_i$  to  $P$ . The gradient of  $\rho_i$  at the point  $\eta_{X_i}(s)$  (where  $0 \leq s \leq r$ ) is

$$(2.2) \quad \nabla \rho_i(\eta_{X_i}(s)) = \eta_{X_i}'(s).$$

Set  $f = \rho_1 - \rho_2$ . Thus the zero set of  $f$  is the set of points that are at equal distances from  $c_1$  and  $c_2$ . This is a  $C^1$  function in a neighborhood of  $P$ . The gradient of  $f$  at  $P$  is

$$(2.3) \quad \nabla f(P) = \nabla \rho_1(P) - \nabla \rho_2(P) = \eta_{X_1}'(r) - \eta_{X_2}'(r)$$

which is not zero (if it were, then the minimizing geodesics from  $P$  to  $\partial M$  would be equal). Therefore by the implicit function theorem the set  $\mathcal{S}$  defined by  $f = 0$  is a  $C^1$  curve in some small open disk  $U$  about  $P$ . The points of  $\mathcal{S}$  are all cut points as they can be connected to  $\partial M$  by two minimizing geodesics. Moreover no other point of  $U$  can be a cut point as any point of  $U$  is either closer to  $c_1$  than  $c_2$  or the other way around. Thus  $\mathcal{C} \cap U = \mathcal{S}$ . As  $\mathcal{S}$  is a level set of  $f$  its tangent at  $P$  is orthogonal to  $\nabla f = \nabla \rho_1 - \nabla \rho_2$ . But each  $\nabla \rho_i$  is a unit vector so that  $\nabla \rho_1 + \nabla \rho_2$  is orthogonal to  $\nabla f$ . But  $\nabla \rho_1 + \nabla \rho_2$  bisects the angle between  $\nabla \rho_1$  and  $\nabla \rho_2$ . This completes the proof of case 1.

If  $k \geq 3$  then choose a sector and reorder things so that this sector is  $S_1$  and so that  $X_1, X_2$  are the points of  $\{X_1, \dots, X_k\} = \partial B_r(p) \cap \partial M$  that are on  $S_1$ . For each  $i$  with  $1 \leq i \leq k$  choose a small piece  $c_i$  of  $\partial M$  centered at  $x_i$  and as in the case of  $k = 2$  let  $\rho_i$  be the distance from  $c_i$ . As before each  $\rho_i$  is  $C^1$  in a neighborhood on the minimizing geodesic segment from  $X_i$  to  $P$ . Again let  $f = \rho_1 - \rho_2$ . Again as before  $\nabla f \neq 0$  near  $P$  and so in some small open disk  $U$  about  $P$  the set defined by  $f = 0$  is a  $C^1$  curve through  $P$  and the tangent to this curve bisects the angle between the two sides of the sector  $S_1$ . Call this curve  $\mathcal{S}$ . Set

$$\mathcal{S}_+ = \mathcal{S} \cap S_1, \quad \mathcal{S}_- = \mathcal{S} \setminus S_1.$$

If  $Q$  is a point in the interior of  $U \cap S_1$ , and the disk  $U$  is small enough, then the point of  $\partial M$  closest to  $Q$  will be in  $c_1$  or  $c_2$ . To see this let  $3 \leq i \leq k$ . Then by the argument above the set  $F_{i1}$  of points in  $U$  closer to  $c_i$  than to  $c_1$  is separated from the set  $G_{i1}$  of points in  $U$  closer to  $c_1$  than to  $c_i$  by a smooth curve  $\gamma_{1i}$  whose tangent at  $P$  bisects the angle between the geodesic segments  $[P, X_i]$  and  $[P, X_1]$ . Likewise for the sets  $F_{2i}$  and  $G_{2i}$ . Therefore the set of points closer to  $c_i$  than either  $c_1$  or  $c_2$  is contained in  $F_{i1} \cap F_{i2}$  and this set is disjoint from  $S_1$ , at least when  $U$  is small enough. This implies the statement above about  $Q$ . But this makes it clear that the part of  $\mathcal{C}$  in  $S_1$  is just  $\mathcal{S}_+$ . As a similar argument applies to the other sectors. This completes the proof.  $\square$

We now provide the details of the proof of Proposition 2.1. First we need that  $M$  and the cut locus  $M$  have very similar topology.



**Proposition 2.6.** *For any compact two dimensional Riemannian manifold  $(M, \partial M)$  the cut locus  $\mathcal{C}$  is a strong deformation retract of  $M$ .*

*Proof.* This is standard. Retraction is accomplished by normal deformation.  $\square$

**Lemma 2.7.** *If  $M$  is simply connected then the cut locus  $\mathcal{C}$  contains at least one focal point.*

*Proof.* Assume, toward a contradiction, that  $\mathcal{C}$  has no focal points. Then by the structure theorem  $\mathcal{C}$  is a graph in the sense that it is a finite number of points connected by a finite number of  $C^1$  imbedded arcs. (Note that loops, that is arcs that begin and end at the same point, may be possible.) Also by the structure theorem each vertex of the graph has valence at least 3 in the sense that there are at least three arcs ending at the vertex. But by Proposition 2.6,  $\mathcal{C}$  has the same homotopy type as  $M$  and thus it is also simply connected. Therefore it is a tree. But a tree has at least two vertices that are “ends” in that they have valence one. This contradiction completes the proof.  $\square$

*Remark 2.8.* Thus we have established Proposition 2.1. In fact, a nontrivial cut locus  $\mathcal{C}$  must have at least two focal points. To see this note that an easy variant of the structure theorem shows that if the set of focal points is finite, then  $\mathcal{S}$  is a graph. Again by the structure theorem any valence one vertex must be a focal point. This if the domain is simply connected there are three cases:

- 1:  $\mathcal{S}$  is a one point set in which case the domain is a disk,
- 2:  $\mathcal{S}$  has more than one point and a finite number of focal points. In this case  $\mathcal{S}$  is a graph and thus has two or more vertices, which implies it has two or more focal points,
- 3:  $\mathcal{S}$  contains an infinite number of focal points.

Note that we are only assuming a one sided bound on  $\kappa$ , not a bound on  $|\kappa|$ . The argument applies equally well to Riemannian surfaces. In the Riemannian surface case, the focal distance depends on the upper bound of the curvature  $K_0$  as well as the boundary curvature.

**Theorem 2.9.** *Let  $M$  be simply connected with  $\partial M \subset \mathcal{K}^+$ . Suppose that the Gauss curvature  $K$  of  $M$  satisfies  $K \leq 0$  and the geodesic curvature  $\kappa$  of  $\partial M$  with respect to the inward normal satisfies  $\kappa \leq 1$  a.e. Then  $M$  contains a disk of radius one.  $M$  has area at least  $\pi$  with equality if and only if  $M$  is isometric with the standard unit disk in the plane.*

*Proof.* As in Lemma 2.7 the cut locus  $\mathcal{C}$  has at least one focal point  $P$ . By adapting standard comparison theorems this focal point has a distance of at least 1 from  $\partial M$ . Also (see, e.g., [K]) the area inequality holds. This implies the theorem.  $\square$

*Remark 2.10.* For general curvature, the radius estimate of the contained disk has the following form. Let  $r_0 > 0$  and let  $K_0$  be any real number. When  $K_0 > 0$  we

assume  $r_0 < \pi/\sqrt{K_0}$ . Set

$$(2.4) \quad \left\{ \begin{array}{ll} \sqrt{K_0} \cot(\sqrt{K_0} r_0), & \text{if } K_0 > 0, \\ \frac{1}{r_0}, & \text{if } K_0 = 0, \\ \sqrt{|K_0|} \coth(\sqrt{|K_0|} r_0), & \text{if } K_0 < 0. \end{array} \right.$$

(2.5)

(2.6)

Then it is straightforward to modify the proof of the last theorem to show that if  $M$  is simply connected, the Gauss curvature of  $M$  satisfies  $K \leq K_0$  and the geodesic curvature of  $\partial M$  with respect to the inward normal satisfies  $\kappa \leq \kappa_0$  *a.e.*, then  $M$  contains a disk of radius  $r_0$ . This gives the area of a disk of radius  $r_0$  in the model space of constant curvature  $K_0$  as a lower bound for the area of  $M$  with equality if and only if  $M$  is isometric to a disk of radius  $r_0$  in the model.

There is an application to minimal (zero mean curvature) surfaces spanning curves in space that follows easily from what we have done.

**Corollary 2.11.** *Let  $c \in \mathcal{K}$  be a curve in  $\mathbf{R}^3$  whose curvature as a space curve has  $k \leq 1$  *a.e.* Assume that  $X : \bar{D} \rightarrow \mathbf{R}^3$  is a minimal immersion of a disk with boundary so that the restriction  $X|_{\partial D}$  is a regular parameterization of  $c$  and  $X \in C^{1,1}(\bar{D}, \mathbf{R}^3)$ . Then the area  $|D| \geq \pi$ .*

*Proof.* Intrinsically  $D$  is a surface with nonpositive Gauss curvature and with boundary geodesic curvature  $\kappa \leq 1$  *a.e.* since the geodesic curvature of  $\partial D = c$  does not exceed curvature of  $c$  viewed as a space curve. Thus Theorem 2.9 shows the  $\text{area}(D) \geq \pi$  with equality if and only if  $D$  is a flat round disk.  $\square$

**Corollary 2.12.** *Let  $c \in \mathbf{R}^3$  be a closed embedded  $C^{1,1}$  space curve whose curvature  $k \leq 1$  at all points. Then any disk spanning  $c$  has area at least  $\pi$ .*

*Proof.* For a  $C^1$  space curve let  $A(c)$  be the infimum of the areas of the disks spanning  $c$ .  $A$  is a continuous function of  $c$  in the  $C^1$  topology. As the real analytic curves are dense in the space of all curves in the  $C^2$  topology we may assume that  $c$  is real analytic. If  $M$  is the Douglas-Rado solution to the Plateau problem, then the area of  $M$  is  $A(M)$  and by a theorem of Osserman [O] and Gulliver [G] this is free of interior branch points, and by a theorem of Gulliver and Lesley [GL] it is free of boundary branch points. Therefore the last corollary implies that  $A(c) \geq \pi$ .  $\square$

*Remark 2.13.* Minimizing surfaces in  $\mathbf{R}^n$  for  $n \geq 4$  can have branch points. We wonder if there is still get a lower bound on the inradius of a disk with a metric that is smooth except for a finite number of singularities of “branch point type”.

*Remark 2.14.* Note that if  $c$  is a space curve that is very close to a standard circle double covered, then there is a minimal surface of the type of a Möbius strip that

spans  $c$  and has small area. However, no lower bound is to be expected if, as for the Möbius strip, the Euler characteristic vanishes. For example if  $M$  is the planar region between concentric circles, one of radius 1 and the other of radius  $1 + 2r$ , then the curvature of boundary of  $M$  has  $|\kappa| \leq 1$  but the largest disk that can be put in  $M$  has radius  $r$ . Since  $r$  can be taken to be as small as we please this shows there is no lower bound for the inradius.

For higher connectivity, there is another inradius lower bound. The following is a special case of a more general theorem of Alexander and Bishop [AB] for curved surfaces under curvature bounds. The proof for the case of plane domains follows easily from what we have already done so we include it here for completeness.

**Theorem 2.15.** *Let  $M \subset \mathbf{R}^2$  be a bounded connected domain in the Euclidean plane with boundary of class  $\mathcal{K}$  (so that the curvature satisfies  $|\kappa| \leq 1$  a.e.) If the Euler characteristic  $\chi(M)$  is non-zero, then  $M$  contains a disk of radius  $r_1 = 2/\sqrt{3} - 1 \approx .15470053838$ .*

**Lemma 2.16.** *If a disk  $B_r(p)$  has three points on the boundary so that the unit disks tangent to  $B_r(p)$  at these points are disjoint, then  $r \geq r_1 = 2/\sqrt{3} - 1$ .*

*Proof.* The extremal figure is three unit disks centered at the vertices of an equilateral triangle with sides of length 2 and  $B_r(p)$  the disk centered at the center of the triangle that is tangent to the three larger disks. This disk has radius  $r_1$  given above.  $\square$

*Proof of theorem 2.15.* If the cut locus of  $M$  has a focal point, then  $M$  contains a disk of radius 1 and we are done. Thus assume that the cut locus  $\mathcal{C}$  has no focal points. If  $\mathcal{C}$  has no vertices then by the structure theorem  $\mathcal{C}$  is a  $C^1$  connected curve, and thus a circle. Thus  $\mathcal{C}$  and, by Proposition 2.6, also  $M$  have Euler characteristic zero. Therefore by the hypothesis  $\mathcal{C}$  has a least one vertex, and by the structure theorem the valence of this vertex  $P$  is at least three. Set  $r = \text{dist}(P, \partial M)$  and let  $\{X_1, \dots, X_k\} = \partial B_r(P) \cap \partial M$ .

Consider unit disks  $\{B_1, \dots, B_k\}$  exterior to  $M$  and tangent to  $\partial M$  at the  $X_j$ . We now argue that these unit disks are disjoint. Suppose this is not the case for, say, two consecutive contact points  $X_1, X_2$ . Then the two tangent unit disks  $B_1$  and  $B_2$  intersect. Let the piece of boundary curve from  $X_1$  to  $X_2$  be denoted  $c(s)$  where  $c(0) = X_1$  and  $c(L) = X_2$ . We first claim that  $L \leq \pi$ . By Proposition 1.1 (1) and (2) there are  $0 < s_1, s_2 < \pi/2$  so that  $c(s_1) \in \bar{B}_2$  and  $c(L - s_2) \in \bar{B}_1$ . Hence  $c([0, s_1]) \cap c([L - s_2, L]) \neq \emptyset$ , therefore  $L \leq s_1 + s_2 \leq \pi$ . By Proposition 1.2, (1) is this is impossible. By Lemma 2.16 this implies  $r \geq r_1$ .  $\square$

*Remark 2.17.* In higher dimensions, Lagunov [L] shows that the largest ball contained in a domain of  $\mathbf{R}^n$ ,  $n \geq 3$ , with connected boundary with principal curvatures satisfying  $|\kappa_i| \leq 1$  has radius at least  $r_1 = 2/\sqrt{3} - 1$  but gives an example for which this cannot be enlarged. However, for domains satisfying additional topological restrictions, such as the ball, Lagunov and Fet [LF] show that the least radius of the

contained ball is increased to  $r_2 = \sqrt{3/2} - 1$  and give examples of surfaces showing this is sharp.

The essence of Lagunov's argument, again, is to study the cut set. Let  $\mathcal{C}(X_1, \dots, X_k)$  be the cut locus for finitely many points  $\{X_1, \dots, X_k\} \in \partial B_1(0)$  on the surface of the unit ball in  $\mathbf{R}^n$ .  $\mathcal{C}(X_1, \dots, X_k)$  divides  $\mathbf{R}^n$  into sectors  $S_i$  containing  $X_i$  where  $S_i$  can be defined as the set of points of  $\mathbf{R}^n$  that are closer to  $X_i$  than to any of the other points in the set  $\{X_1, \dots, X_k\}$ . If  $k = 2$  then  $\mathcal{C}(X_1, X_2)$  is the hyperplane that is a perpendicular bisector of the segment between  $X_1$  and  $X_2$ . For a domain  $M$  with smooth boundary  $\partial M$  let  $\mathcal{C}$  be the cut locus of  $\partial M$ . Choose  $P \in \mathcal{C}$  such that  $P$  is not a focal point of  $\partial M$ . Set  $r = \text{dist}(P, \partial M)$ . Lagunov shows that there is a finite number  $k \geq 2$  of points  $\{X_1, \dots, X_k\}$  in  $\partial B_r(P) \cap \partial M$ . For each  $i$  let  $\mathbf{u}_i$  be the unit vector in  $T_P M$  that is tangent to the segment  $[P, X_i]$  and is pointing in the direction of  $X_i$ . If  $k = 2$  then there is a small open ball  $U$  about  $P$  so  $U \cap \mathcal{C}$  is a smooth hypersurface and the tangent space to  $\mathcal{C}$  at  $P$  is  $\mathcal{C}(\mathbf{u}_1, \mathbf{u}_2)$ . If  $k \geq 3$  then there is a small ball  $U$  about  $P$  so that  $U \cap \mathcal{C}$  is a "nice" stratified set, in particular it has a tangent cone at  $P$  and this tangent cone is  $\mathcal{C}(\mathbf{u}_1, \dots, \mathbf{u}_k)$ . For general domains, if there is no focal point, the cut set must contain points where  $k \geq 3$ . Locally this looks like a triple junction graph crossed with an interval, hence it allows balls of radius  $r_1$  as in Theorem 2.15. Under topological hypotheses, Lagunov and Fet [LF] deduce existence  $k = 4$  points which yields the larger radius  $r_2$  in the analog of Lemma 2.16.

There may be a stronger inradius estimates in the higher dimensional version if topological assumptions are replaced by geometric ones such as a bound on diameter. Also, we suspect that any starlike domain in  $\mathbf{R}^3$  with all principle curvatures  $\leq 1$  has inradius  $\geq 1$ .

### 3. Gradient estimate and star-shapedness

The following lemma gives an estimate on the radial component of velocity of a curve in an annulus in the plane. It says that the rate at which a curve approaches the boundary circles cannot be too large near the boundary lest the curve be forced to "drive into the curb". The estimate was found by computing the gradient of the distance function on a circular arcs tangent to the bounding circles.

**Lemma 3.1.** *Suppose  $M \subset \mathbf{R}^2$  is an embedded disk with class  $\mathcal{K}$  boundary satisfying  $|\kappa| \leq 1$  a.e. Let  $c(s)$  denote the boundary curve parameterized by arclength and  $\rho(s) = \langle c(s), c(s) \rangle$  be the square of the distance to the origin. If the unit disk centered at the origin  $B_r(0) \subset M$  then*

$$(3.1) \quad \rho_s^2 \leq [\rho - r^2] [(r + 2)^2 - \rho] \quad \text{whenever } \rho \leq r^2 + 2r.$$

If  $M \subset B_R(0)$  where  $R \leq 2$  then

$$(3.2) \quad \rho_s^2 \leq [R^2 - \rho] [\rho - (R - 2)^2].$$

Moreover, inequality (3.2) implies that  $\rho \geq (2 - R)^2$ . If  $2 < R$  then (3.2) holds whenever  $\rho \geq R^2 - 2R$ .

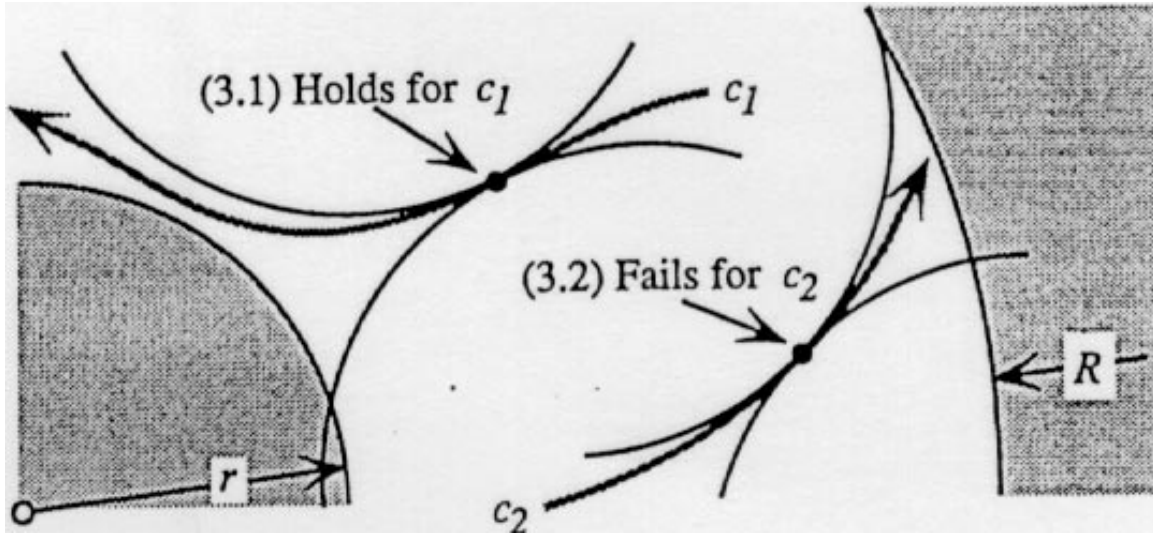


Fig. 2 A curve with large  $|\rho_s|$  can't avoid "driving into the curb".

*Proof.* For (3.1), we may assume that  $c'(0)$  makes an angle of at most 90 degrees with the segment from  $c(0)$  to the origin. (If not replace  $c(s)$  with  $c(-s)$ .) Let  $C_1$  and  $C_2$  be two unit circles tangent to  $c$  at  $c(0)$  with  $C_1$  the one that is "closest" to the origin. If  $\rho \leq r^2 + 2r$  and (3.1) is false, then both  $C_1$  and  $C_2$  must intersect the interior of  $B_r(0)$ . Proposition 1.1 implies that  $c$  will also intersect the interior of  $B_r(0)$ , which is not the case. Thus  $C_1$  is either tangent to  $\partial B_r(0)$  or disjoint from the closure of  $B_r(0)$ . This implies that if the angle between  $c'(0)$  and the segment from  $c(0)$  to the origin is at least as large as the angle between this segment and the tangent to the unit circle through  $c(0)$  and tangent to  $\partial B_r(0)$ . A calculation shows that this is equivalent to

$$(3.3) \quad [\rho'(s)]^2 = [(r + 1)^2 - \rho(s)][\rho(s) - r^2].$$

(3.2) can be proven along the same lines, however we indicate an alternate proof. As before,  $\rho_s$  is absolutely continuous and  $\rho_{ss}$  is defined almost everywhere. The (weak) equations for  $\rho$  on  $\partial M$  are

$$(3.4) \quad \begin{aligned} \rho_s &= 2\langle c, \mathbf{t} \rangle \\ \rho_{ss} &= 2 - 2\kappa p \quad a.e. \end{aligned}$$

where  $p = -\langle X, \mathbf{n} \rangle$  and  $\mathbf{t}$  and  $\mathbf{n}$  are the tangent and outer unit normal vectors to  $c$ . Thus setting  $\rho_s^2 + 4p^2 = 4\rho$  we get

$$(3.5) \quad \rho_{ss} \leq 2 + \sqrt{4\rho - \rho_s^2}. \quad a.e.$$

This can be integrated to yield (3.2). □

**Lemma 3.2.** *Suppose  $M \subset \mathbf{R}^2$  is an embedded disk with boundary of class  $\mathcal{K}(k)$ , that is, the boundary curve  $c(s)$  is  $C^1$ , parameterized by arclength, with  $c'$  absolutely continuous with boundary curvature  $|\kappa| \leq k$  a.e. Let  $p = -\langle c, \mathbf{n} \rangle$  denote the support function to  $\partial M$  and assume  $p \geq p_0 > -1/k$  at all  $x \in \partial M$ . Then*

$$(3.6) \quad L \leq \frac{2kA + 2\pi p_0}{1 + kp_0}.$$

*In particular, if  $M$  were star-shaped with respect to the origin (thus  $p_0 \geq 0$ ) then  $L \leq 2kA$ .*

*Proof.* This follows from the Minkowski formulas whose proof we sketch. By Lebesgue's theorem we can recover  $\rho_s$  by integration. (3.4) implies

$$0 = \int_{\partial M} \rho_{ss} ds = 2 \int_{\partial M} ds - 2 \int_{\partial M} \kappa p ds$$

so that

$$L = \int_{\partial M} \kappa p ds.$$

The function  $f(X) = \langle X, X \rangle$  satisfies  $\mathbf{n}f = 2\langle c, \nabla_{\mathbf{n}}c \rangle = 2p$  for  $X \in \partial M$ . Using  $p \geq p_0$  and Stokes theorem,

$$\begin{aligned} L - 2\pi p_0 &= \int_{\partial M} \kappa (p - p_0) ds \\ &\leq k \int_{\partial M} p - p_0 ds \\ &= \frac{k}{2} \int_{\partial M} \frac{\partial f}{\partial N} ds - kp_0 L \\ &= \frac{k}{2} \int_M \Delta f dx \wedge dy - kp_0 L \\ &= 2kA - kp_0 L \end{aligned}$$

where  $\Delta$  is the  $\mathbf{R}^2$  Laplacian. Hence (3.6) holds.  $\square$

*Remark 3.3.* In fact, there is a reverse isoperimetric inequality in all dimensions for regions  $M \subset \mathbf{R}^n$  which are starlike with respect to the origin and with mean curvature  $H$  (normalized so that  $H = 1$  on the unit sphere) of  $\partial M$  satisfying

$|H| \leq 1$ . To see this let  $p$  the support function of  $\partial M$ . Then the surface area  $A$  of  $\partial M$  and the volume  $V$  of  $M$  are given by the Minkowski formulas

$$A = \int_{\partial M} pH \, dA, \quad nV = \int_{\partial M} p \, dA.$$

Using that  $|H| \leq 1$  and that  $p \geq 0$  (as  $M$  is starlike with respect to the origin) in these leads at once to the reverse isoperimetric inequality

$$A \leq nV.$$

Also the isoperimetric inequality in  $\mathbf{R}^n$  is

$$A(\mathbf{S}^{n-1})^n V(M)^{n-1} \leq V(B^n)^{n-1} A(\partial M)^n.$$

Using the relation  $A(\mathbf{S}^{n-1}) = nV(B_1^n)$  and the last two inequalities leads to sharp lower bounds for the  $A(\partial M)$  and  $V(M)$

$$A(\mathbf{S}^{n-1}) \leq A(\partial M), \quad V(B^n) \leq V(M),$$

in the class of star like regions with mean curvature having  $|H| \leq 1$ .

The following lemma is the link between radius bounds and length bounds. Its main conclusion is star shapedness of the domain.

**Theorem 3.4.** *Suppose  $M \subset \mathbf{R}^2$  is an embedded disk with  $\partial M$  of class  $\mathcal{K}$ . Suppose also that  $B_1(0) \subset M \subset B_R(0)$  where  $R \leq 3$ . Then  $M$  is star shaped with respect to the origin. There is the estimate*

$$(3.7) \quad L \leq \frac{8A + 2\pi(3 + 2R - R^2)}{7 + 2R - R^2}$$

where  $L = |\partial M|$  is the length and  $A = |M|$  is the area of  $M$ . In particular, for all  $1 \leq R \leq 3$  there holds  $L \leq 2A$ .

*Proof.* If  $R < 3$  then Lemma 3.1 with  $r = 1$  and  $R = 3$  imply that  $\rho_s^2 < 4\rho$  and so  $4\rho^2 = 4\rho - \rho_s^2 > 0$  so  $p$  cannot change sign and must remain positive. If  $R = 3$  the only possibility for  $p < 0$  is that at a point  $c(s_0) \in \partial M$  there holds  $p = 0$  at  $\rho = 3$  and  $\partial M$  consists of circular arcs near  $c(s_0)$  extending the entire range  $\rho \in [1, 9]$ . But at  $\rho \in \{1, 9\}$  we must have  $p = +\sqrt{\rho}$  and  $P \geq 0$  on the arcs adjacent to  $c(s_0)$  or else  $M$  is on the “wrong” side of  $\partial M$  so at  $\rho = 1$  and  $1 \neq \inf \rho$ , or  $\rho = 9$  and  $9 \neq \sup \rho$ . The remainder of the argument is to give an estimate of  $p_0$ . Recall that

$$(3.8) \quad 4p^2 = 4\rho - \rho_s^2.$$

For  $R < 3$  the bounds given by Lemma 3.1 intersect at  $\rho = \rho_3 := (3 - 2R + R^2)/2$ . Hence, there is an absolute bound

$$\rho_s^2 \leq (9 - \rho_3)(\rho_3 - 1) = \frac{(5 - R)(3 + R)(1 - R)^2}{4}.$$

Observe that

$$4\rho - \min \{(\rho - 1)(9 - \rho), [R^2 - \rho] [\rho - (R - 2)^2]\}$$

is minimized at  $\rho_3$ . Thus by (3.8) we get the bound

$$4p^2 \geq 4\rho_3 - (\rho_3 - 1)(9 - \rho_3) = \frac{(3 - R)^2(1 + R)^2}{4}$$

hence

$$p_0 \geq \frac{(3 - R)(1 + R)}{4}$$

which gives (3.7) when inserted in Lemma 3.2.  $\square$

For  $0 \leq \delta \leq \sqrt{3}$ , let  $E_\delta = B_1((0, \delta)) \cup B_1((0, -\delta))$  denote the domain consisting of the union of two unit disks whose centers are  $2\delta$  apart. Consider two unit circular arcs  $c_1, c_2$  tangent to each of the circular boundary components of  $\partial E_\delta$ . Let  $P_\delta$  denote the ‘‘peanut shaped’’ domain consisting of  $P_\delta = \text{int}(\bar{E}_\delta \cup \bar{F}_1 \cup \bar{F}_2)$  where the  $F_i$  are ‘‘fillets’’ consisting of the region between the arcs  $c_i$  and  $E_\delta$ . This is the region pictured in Figure 1.

**Lemma 3.5.** *Suppose  $\delta \leq 1$ . Let  $\gamma(s) : [0, L] \rightarrow \mathbf{R}^2$  be a curve of regularity  $\mathcal{K}$  with curvature bounded by  $|\kappa| \leq 1$  a.e. whose endpoints are on  $\partial P_\delta$  and which lies exterior to  $E_\delta$ . Then the entire curve is exterior to  $P_\delta$ .*

*Proof.* Suppose that  $c_1$  is an arc of the circle  $\partial B_1(\sqrt{4 - \delta^2}, 0)$  and that  $\gamma(s)$  enters the fillet  $F_1$ . Then  $f(x, y) = (x - \sqrt{4 + \delta^2})^2 + y^2 = 1$  on the endpoints of  $\gamma(s)$  and maximal on  $\gamma$  at  $0 < s_0 < L$  where  $f(\gamma(s_0)) > 1$ . Consider the family of unit circle arcs corresponding to circles with centers on the line  $(\sqrt{4 - \delta^2}, 0)$  to  $\gamma(s_0)$ . One of them passes through and is tangent to  $\gamma$  at  $s_0$ . By construction, these are transverse to  $\partial E_\delta$  and hence by the maximum principle,  $\gamma$  ‘‘crashes’’ into  $\partial E_\delta$  before it reaches  $c_1$ .  $\square$

**Lemma 3.6.** *Suppose  $M$  is an embedded disk whose boundary curve is of class  $\mathcal{K}$ . Assume that  $M$  contains two unit disks whose centers  $Z_1, Z_2$  lie on the cut locus  $\mathcal{C}$  of  $\partial M$ . If  $\text{dist}(Z_1, Z_2) = 2\delta > 2$  then*

$$(3.9) \quad |M| > \pi + 2\sqrt{3}.$$



*Proof.* Since it is connected, the cut locus connects  $Z_1$  to  $Z_2$ . We may suppose that  $\mathcal{C} - (B_1(Z_1) \cup B_1(Z_2))$  contains no focal points  $\zeta$ . Otherwise  $M$  contains a unit disk about  $\zeta$  and

$$|B_1(Z_1) \cup B_1(Z_2) \cup B_1(\zeta)| \geq \pi + 2 + \frac{2\pi}{3} > \pi + 2\sqrt{3}$$

and we are done. Hence we suppose that there is a piecewise  $C^1$  simple subarc  $\gamma : [0, L] \rightarrow \mathcal{C}$ , connecting  $Z_1 = \gamma(0)$  to  $Z_2 = \gamma(L)$  so that  $\gamma([0, L]) - (B_1(Z_1) \cup B_1(Z_2))$  is without focal points.

We will show that either  $M$  contains a peanut  $P_\delta$  or it contains thickening where  $\gamma$  meets the disks. In order to quantify this, consider three disks  $B = B_1(0, 0)$ ,  $B' = B_1(\sqrt{3}, 1)$  and  $B'' = B_1(\sqrt{3}, -1)$ . The triangular region  $T$  between the three circles has area  $\sqrt{3} - \pi/2$ . Let  $\sigma(x)$  denote the radius of the circle centered at  $(x, 0)$  and tangent to  $B'$  and  $B''$ . Let  $\sigma(x) = 0$  if  $x \geq \sqrt{3}$ . Let  $\tau(x)$  denote the distance of the contact point of  $\partial B_{\sigma(x)}(x, 0) \cap \partial B'$  to the origin and  $\lambda(\tau(x))$  the length of the arc  $\partial B_{\tau(x)}(x, 0) \cap T$ .  $\sigma(x)$  is a decreasing and  $1 \leq \tau(x) \leq \sqrt{3}$  an increasing function of  $x$ . We say that  $\gamma$  has *remote ends* if for all  $0 \leq x_1, x_2$  there are measurable functions  $t_1(x_1), t_2(x_2) \in [0, L]$  so that  $\text{dist}(\gamma(t_1(x_1)), Z_1) = x_1$ ,  $\text{dist}(\gamma(t_2(x_2)), Z_2) = x_2$  and that  $B_{\sigma(x_1)}(\gamma(t_1(x_1))) \cap B_{\sigma(x_2)}(\gamma(t_2(x_1))) = \emptyset$ .

First, if  $\gamma$  has remote ends then  $|M| > \pi + 2\sqrt{3}$ . To see this, observe that for any point  $\gamma(t)$  there is an estimate of cut distance  $c(\gamma(t))$ . In particular, let  $X_1, X_2 \in \partial M$  be two distinct points with  $c(\gamma(t)) = |c(t) - X_i|$ , which by the structure of  $\mathcal{C}$  may be chosen on opposite sides of  $\gamma$ . The subsets of  $\partial M$  which are within an arclength  $\pi/2$  of the  $X_i$  must be disjoint by Proposition 1.2. They must also be disjoint of  $B_1(Z_1)$ . Hence, by Proposition 1.1, the smallest cut distance is possible if the boundary subsets are unit circle arcs and form with  $\partial B_1(Z_1)$  a triangle congruent to  $T$ . Thus the cut distance is at least that of the distance from  $(|Z_1 - \gamma(t)|, 0)$  to  $\partial T$ , namely  $c(\gamma(t)) \geq \sigma(|\gamma(t) - Z_1|)$ . In particular, there is an arc

$$\mathcal{A}_1(\tau(|Z_1 - \gamma(t)|)) = \partial B_{\tau(|Z_1 - \gamma(t)|)}(Z_1) \cap B_{\sigma(|\gamma(t) - Z_1|)}(\gamma(t)) \subset M.$$

There is a similar definition of an arc  $\mathcal{A}_2$  near the  $Z_2$  end.

If  $\gamma$  has remote ends, then for all  $0 \leq x_i < \sqrt{3}$  the balls  $B_{\sigma(|\gamma(t_1) - Z_1|)}(\gamma(t_1))$  and  $B_{\sigma(|\gamma(t_2) - Z_2|)}(\gamma(t_2))$  are disjoint, as are the corresponding arcs  $\mathcal{A}_i(\tau_i)$ . Hence the thickenings

$$\mathcal{H}_i = \bigcup_{1 < \tau < \sqrt{3}} \mathcal{A}_i(\tau)$$

are disjoint. Thus

$$\begin{aligned}
 (3.10) \quad |M| &\geq |B_1(Z_1)| + |B_1(Z_2)| + |\mathcal{H}_1| + |\mathcal{H}_2| \\
 &= 2\pi + 2 \int_1^{\sqrt{3}} \lambda(\tau) d\tau \\
 &= 2\pi + 2|T| \\
 &= \pi + 2\sqrt{3}.
 \end{aligned}$$

Moreover, since  $\gamma(t_1(\sqrt{3})) \notin \mathcal{H}_1 \cup \mathcal{H}_2$  by continuity, but is an interior point of  $M$ , the inequality (3.10) must be strict. In particular, if  $\delta \geq \sqrt{3}$  then  $\gamma$  has remote ends so (3.9) holds.

From now on, assume  $\delta < \sqrt{3}$ . Let  $\Lambda$  be the straight line segment from  $Z_1$  to  $Z_2$ . By rigid motion we may suppose that  $\Lambda$  is in the  $y$ -axis and centered about 0. We deal with the case that  $\Lambda \not\subset M$ . Choose a point  $E \in \Lambda - \bar{M}$ . We may also suppose that the bounded part of the complement of  $M \cup \Lambda$  lies on the right side of  $\Lambda$ . Let  $\chi : [0, \infty) \rightarrow \mathbf{R}^2$  in the unbounded part of the complement of  $M \cup \Lambda$  be a simple smooth path connecting  $E$  to  $\infty$ . Let  $P_\delta$  denote the peanut shaped region about the balls  $B_1(Z_1)$  and  $B_1(Z_2)$ . Let  $Y_1, Y_2 \in \Lambda$  be the endpoints of the interval of  $\Lambda - \bar{M}$  containing  $E$ . Because of Lemma 3.1, the direction  $V$  to  $\partial M$  at  $X \in \partial M \cap P_\delta$  cannot point at  $Z_1$  nor  $Z_2$  thus must “flow through”. For example, foliate  $\mathbf{R}^2 - M$  by arcs of the circle  $x = +(1 - y^2)^{1/2} + k$  for all constants  $k$  so that the semicircle touches  $B_1(Z_1)$  and  $B_1(Z_2)$ . Then  $k|_{\partial M}$  cannot have a maximum at one of these arcs, by the maximum principle. In particular, the curves  $\partial M$  through  $Y_i$  continue to  $x > 1$ .

Consider the portion  $\beta$  of  $\partial M$  starting  $\pi/2$  from  $Y_1$ , continuing through  $Y_1$ , heading in the positive  $x$ -direction to  $Y_2$  and ending  $\pi/2$  beyond  $Y_2$ . By the Schur-Schmidt Proposition 1.1,  $\beta$  starts and ends on the left side of  $y = -1$  and passes through  $P_\delta$ . Let  $\mathcal{E}$  denote the “lagoon”, the connected component of  $\mathbf{R}^2 - (M \cup \Lambda)$  containing  $E$  bounded by the union of the arc  $\beta$  and the segment  $[Y_1, Y_2]$ . Now consider the cut set  $\mathcal{C}'$  of the complement of  $M$ . Since there are points  $(1, y) \in \mathcal{E}$ , some point on  $\mathcal{E}$  is a cut point of  $\mathbf{R}^2 - M$ . To see this, let  $B_u(Y)$  be the largest ball contained in  $\mathbf{R}^2 - M$  with center  $Y \in [Y_1, Y_2]$ . It must be a cut point of  $\mathcal{C}'$  which is not a focal point because, by construction,  $u < \sqrt{3} - 1$ . Let  $\{X_1, \dots, X_k\}$  be the nearest points of  $Y$  in  $\partial M$ . It must happen that at least one  $X_1$  is on the boundary near  $Y_1$  and one  $X_2$  is on the boundary near  $Y_2$ . Now  $Y$  is the only place on the two segments  $[X_1, Y] \cup [Y, X_2]$  that meets  $\mathcal{C}'$ . By the structure of the cut set, at least one continuation of the cut set enters the region  $\hat{\mathcal{E}}$  bounded by  $[X_1, Y] \cup [Y, X_2] \cup \beta$ . But  $\hat{\mathcal{E}}$  is simply connected, therefore by the structure theorem of  $\mathcal{C}'$ , if there are no focal points  $\hat{\mathcal{E}} \cap \mathcal{C}'$  is a piecewise  $C^1$  tree that must have an endpoint other than  $Y$ . It is a focal point  $W$  and therefore  $B_1(W) \cap M = \emptyset$ . In particular, the intrinsic

minimal path from  $Z_1$  to  $Z_2$  within  $\mathbf{R}^2 - \left(\chi([0, \infty)) \cup \hat{\mathcal{E}}\right)$  must loop around  $B_1(W)$  so must have a length at least  $2\sqrt{3} + 4\pi/3 - 2\sin^{-1}(\delta/2) > 2\pi/3 + 2\sqrt{3}$ . Thus,  $\gamma$  has remote ends and (3.9) holds.

Finally, consider the case that  $\Lambda \subset M$ . By considering the foliation of  $P_\delta - (B_1(Z_1) \cup B_1(Z_2) \cup \Lambda)$  by the field of extremals again,  $x = \pm(1 - y^2)^{1/2} \pm k$ , we see that any part of  $\partial M$  that begins and ends outside the peanut  $P_\delta$  must stay outside the peanut. Hence

$$|M| \geq |P_\delta| = \pi + 2\delta\sqrt{4 - \delta^2} > \pi + 2\sqrt{3}$$

whenever  $1 < \delta < \sqrt{3}$ , which is the present case. □

We now prove the first link of our main estimate. It says that among domains with boundary having bounded curvature, the circumradius is bounded by area. In fact we give a “stable version”: the closer the area is to the area of a disk, the closer the domain itself is to the disk.

**Theorem 3.7 [Area Stability Theorem].** *Suppose  $M$  is an embedded disk whose boundary curve is of class  $\mathcal{K}$ . If the area  $|M| \leq \pi + 2\sqrt{3}$  then there is a point  $P \in M$  so that  $B_1(P) \subset M \subset B_R(P)$  where*

$$(3.11) \quad (R - 1)^2 = 8 - 2\sqrt{16 - (A - \pi)^2}.$$

*Proof.* We utilize the structure of the cut locus  $\mathcal{C}$  of  $M$ . By Proposition 2.1,  $M$  contains at least one unit disk which we locate at the origin. Let  $\zeta(A) = R$  be the radius defined by the relation (3.11) for  $\pi \leq A \leq \pi + 2\sqrt{3}$ .  $\zeta(A)$  is the radius needed to contain a peanut located so the origin coincides with one of the unit disks of the peanut. So  $B_1(0) \subset P_\delta \subset B_{\zeta(A)}(0)$  for a given area  $A = |P_\delta|$ . Denote by

$$\mathcal{G} = \{P \in \mathcal{C} : \text{dist}(P, \partial M) + |P| \leq \zeta(A)\}.$$

We claim  $\mathcal{G} = \mathcal{C}$  and so  $R \leq \zeta(A)$ . First observe that if  $Q \in \mathcal{C}$  and  $\text{dist}(Q, \partial M) \geq 1$  then  $Q \in \mathcal{G}$ . This implies that there is a unit ball  $B_1(P) \subset M$  so that  $|P| = |Q| + \text{dist}(Q, \partial M) - 1$ . However Lemma 3.6 and  $|M| \leq A$  implies that  $|P| \leq \zeta(A) - 1$  hence  $Q \in \mathcal{G}$ . Moreover, there is a segment from the origin,  $[0, Q]$ , which is completely contained in  $\overline{M}$ . Hence all the focal points of  $\mathcal{C}$ , which have  $\text{dist}(Q, \partial M) \geq 1$ , are in  $\mathcal{G}$ . Thus  $\mathcal{C} - \mathcal{G}$  is a subset of a tree consisting of  $C^1$  curves joined at vertices with finite valence.

Suppose  $\mathcal{C} \neq \mathcal{G}$ . Then at some point  $X(s_0) \in \partial M$  we have  $|X(s_0)| > \zeta(A)$ . Consider the function  $f(X) = |X|$  restricted to  $\mathcal{C}$ . At its maximum  $Y \in \mathcal{C}$ , it exceeds  $|Y| \geq |X(s_0)| - |X(s_0) - C(s_0)| > \zeta(A) - 1 > 0$ . Because of the structure of the cut locus, there are  $2 \leq k < \infty$  points  $\{X_1, \dots, X_k\} \subset \partial M \cap \partial B_r(y)$  where  $r = \text{dist}(Y, \partial M) < 1$ . For any one of the components  $(X_i, X_{i+1}) \subset \partial M - \{X_1, \dots, X_k\}$ ,

the cut set bisects the sector and extends beyond  $Y$  in the direction from  $Y$  to  $\frac{1}{2}(X_i + X_{i+1})$ . But because  $f$  is maximal at  $Y$  this implies that  $k = 2$  and the cut set directions are at best perpendicular to  $Y$  with  $Y_1 = (1 + r/|Y|)Y$  and  $Y_2 = (1 - r/|Y|)Y$ . In particular, the segment  $[Y_1, Y_2]$  intersects  $\mathcal{C}$  at exactly one point  $Y$  where it crosses transversally. Let  $\lambda$  be a simple curve in  $\mathbf{R}^2 - M$  connecting  $Y_1$  to  $Y_2$ . Now, each connected component  $\mathcal{C}_i$ ,  $i = 1, 2$ , of  $\mathcal{C} - \{Y\}$  must contain a focal point  $A_i$ . To see this, suppose  $\mathcal{C}_1$  does not. By construction it has the structure of a smoothly embedded connected tree. Hence it must contain at least two vertices of valence 1, thus a focal point. The set  $\mathcal{H} = \mathcal{C} \cup \Gamma_A \cup \gamma - \{Y\} \subset M$  connects both sides the loop  $\lambda \cup [Y_1, Y_2]$  without crossing it, which is a contradiction in  $\mathbf{R}^2$ .  $\square$

**Corollary 3.8. [Preliminary reverse isoperimetric inequality].** *Suppose  $M$  is an embedded disk whose boundary curve is of class  $\mathcal{K}$ . If the area  $|M| \leq \pi + 2\sqrt{3}$  then  $L \leq 2A$ . Moreover  $M$  is star shaped with respect to the center of any unit disk in  $M$ . In fact, if  $M$  is located so that the origin is the center of such a disk, then the rays through the origin are transverse to  $\partial M$  except, possibly, when  $A = \pi + 2\sqrt{3}$  and  $R = \sqrt{8}$ .*

*Proof of Corollary 3.8.* By Theorem 3.7,  $R = R(A) \leq 3$ . The result follows from Theorem 3.4. In fact, the sharper inequality (3.7) holds for this  $R$ .  $\square$

We can extend Theorem 3.4 in the following manner. The closer the circumradius of a domain whose boundary has uniformly bounded curvature is to one, the closer the domain is to the unit circle. We obtain a stability estimate provided the original circumradius is at most the circumradius of three touching circles. The case  $S \leq 2$  also follows from Theorem 3.4.

**Theorem 3.9. [Circumradius stability theorem].** *Let  $M \subset \mathbf{R}^2$  be an embedded disk with boundary of class  $\mathcal{K}$ . Suppose in addition that there is a bound on the circumradius  $M \subset B_S(0)$  where  $1 \leq S \leq 1 + 2/\sqrt{3}$ . Then, in fact,  $B_{s_1}(Z) \subset M \subset B_S(Z)$  where*

$$s_1 = \sqrt{3 + 2S - S^2} - 1$$

and some  $Z \in M$ .  $M$  is star shaped with respect to  $Z$ . Moreover, there is an estimate of the length of the form

$$(3.12) \quad L \leq \frac{4A + 2\pi(1 + 2S - S^2)}{3 + 2S - S^2}.$$

*Proof.* First we show that if two unit disks are in  $M$  then so is the peanut between them. Let  $B_1(Z_1)$  and  $B_2(Z_2)$  two unit disks contained in  $M$ . Let  $\Lambda = [Z_1, Z_2]$  be the line segment between the centers of length  $2\delta$ . Since  $\delta < \sqrt{3}$  by assumption, we

argue as in Lemma 3.6. If  $\Lambda \not\subset M$  then there is a unit disk  $B_1(W) \cap M = \emptyset$  which is contained in the convex hull of  $M$ . However, the circumradius of three tangent unit disks,  $1 + 2/\sqrt{3}$ , is strictly smaller than the circumradius of  $B_1(Z_1) \cup B_1(Z_2) \cup B_1(W)$  which is less than  $S$ . This is a contradiction. Hence  $\Lambda$ , and therefore as in Lemma 3.6, the peanut  $P_\delta \subset M$ .

We may assume that  $B_S$  is the smallest possible disk that contains  $M$  and that its center is at the origin. If we let  $\mathcal{Y} = \partial B_S(0) \cap \partial M$  be the contact points then the origin must be in the convex hull of  $\mathcal{Y}$ . We claim if  $C(Y)$  is a cut point for  $Y \in \mathcal{Y}$  then  $s_0 = |Y - C(Y)| \geq 1$ . The argument is similar to the proof of Lemma 3.6. If not,  $s_0 < 1$  and the cut point is not a focal point. Hence, for  $r(X) = |X|$ , the distance function from the origin, and  $W \in \mathcal{C}$  where  $r|_{\mathcal{C}}$  is maximum,  $|W| \geq S - s_0$  so  $W$  is not a focal point. It follows that the contact points  $\{X_1, \dots, X_k\} = \partial B_{c(W)}(W) \cap \partial M$  must be  $X_1 = (1 + c(W)/|W|)W$  and  $X_2 = (1 - c(W)/|W|)W$ , because by the structure of the cut set near  $W$ ,  $\mathcal{C}$  has maximal  $r$  so must be perpendicular to  $W$  there. Now let  $\gamma$  be a curve in  $\mathbf{R}^2 - M$  connecting  $X_1$  to  $X_2$ . Let  $B_1(Z_1)$  and  $B_2(Z_2)$  be two unit balls centered at focal points on each component of the sets  $\mathcal{C} - \{W\}$ . By the previous paragraph, the line segment  $\Lambda$  from  $Z_1$  to  $Z_2$  is in  $M$ . Hence the connected set  $(\mathcal{C} - \{W\}) \cup \Lambda$  does not intersect the closed path  $[X_2, W] \cup [W, X_1] \cup \gamma$  yet connects its two sides. This is a contradiction.

Finally, since we have shown that the cut distances  $c(Y) \geq 1$  for all  $Y \in \mathcal{Y}$ , there is an osculating unit disk  $D(Y) \subset M$  centered at  $\eta_Y(1)$ . Every segment connecting pairs of  $\eta_Y(1)$ 's for  $Y \in \mathcal{Y}$  must be in  $M$  as must their peanuts. The origin is in the convex hull of the  $\eta_Y(1)$ 's of  $Y \in \mathcal{Y}$  and as the segments connecting these points are in  $M$  which is simply connected, this implies the origin is in  $M$ . To estimate  $\text{dist}(0, \partial M)$ , observe that the origin is in the set consisting of the union of all peanuts with centers at  $\eta_Y(1)$ 's of  $Y \in \mathcal{Y}$ , with any possible holes filled in (union in the bounded components of the complement of the union of peanuts). Hence the minimal distance to the boundary is the distance to the boundary of the narrowest possible peanut, namely one with  $\text{diam}(P_\delta) = 2\delta + 2 \leq 2S$ . Hence  $\text{dist}(0, \partial M) \geq \text{dist}(0, P_{(S-1)}) = s_1 = (3 + 2S - S^2)^{1/2} - 1 \geq \sqrt{8/3} - 1$ .

Thus  $S - s_1 < 2$  and we may continue the argument as in Theorem 3.7. Let  $\rho(X) = \langle X, X \rangle$  be the square of the distance from the origin. We find that the bounds from Lemma 3.1 with  $r = s_1$  and  $R = S$  agree when  $\rho = 1$  so

$$4p^2 = 4\rho - \rho_s^2 \geq 4 - [S^2 - 1] [1 - (S - 2)^2].$$

This lower bound for  $p$  is used in Lemma 3.2 to obtain (3.12). □

**Corollary 3.10.** *Let  $M \subset \mathbf{R}^2$  be an embedded disk with boundary of class  $\mathcal{K}$ . Suppose in addition that there is a bound on the circumradius  $M \subset B_S(0)$  where  $1 \leq S \leq 1 + 2/\sqrt{3}$ . Then*

$$L \leq \frac{2\pi(1 + S)}{3 - S}; \quad L \leq \frac{3A + \pi}{2}.$$

*Proof.* For the first inequality substitute  $A \leq \pi S^2$  into Theorem 3.9. For the second substitute  $S \leq 1 + 2/\sqrt{3}$ .  $\square$

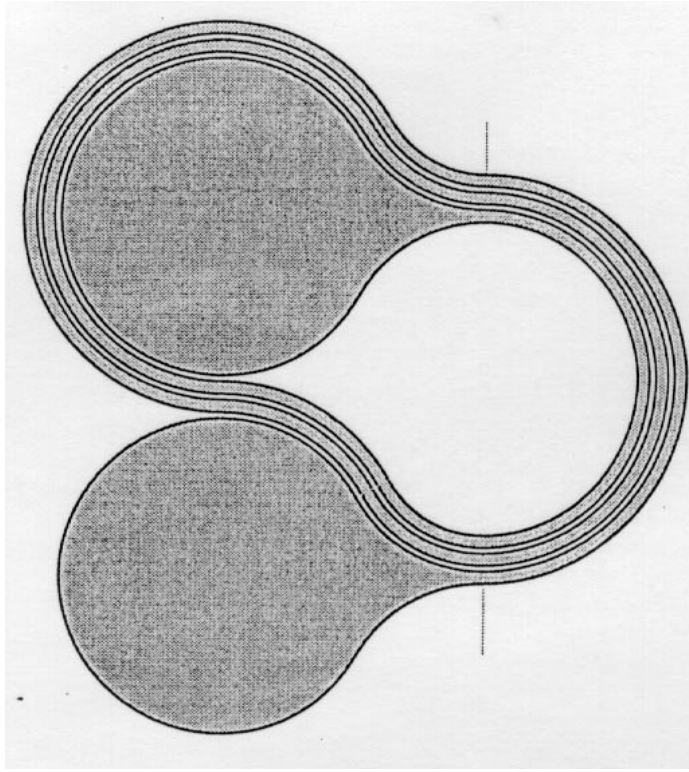


Fig. 3 “Fanbelt” counterexample.

*Remark 3.11.* Observe that Theorem 3.4, Theorem 3.7 and Theorem 3.9 are sharp. That is, for every  $\varepsilon > 0$  there is an example  $M$  that has  $A < \pi + 2\sqrt{3} + \varepsilon$ ,  $B_1(Z) \subset M \subset B_{3+\varepsilon}(Z)$  and circumradius  $S < 1 + 2/\sqrt{3} + \varepsilon$  but has arbitrarily large length. Take three disjoint circles in  $B_{1+2/\sqrt{3}+\varepsilon/2}$  which can be thought of as pulleys. Think of the domain as the region between a long fanbelt (Figure 3). Thread a loop about one of the circles, wind both sides around a second circle, then continue arbitrarily many times about the peanut formed by the first two circles. End by looping about the last circle. By taking the region sufficiently thin, one can obtain all four conditions.

The last result in this section shows that embedded disks with small area but large length have to be thin in a sense appropriate for  $\partial M \in \mathcal{K}$ . In this way we get a quantitative description of the degeneration of embeddedness or “puckering”. A measure of the pinching of a domain is given by

$$(3.13) \quad \omega(M) = \text{dist}(\mathcal{C}, \partial M),$$

where  $\mathcal{C}$  is the cut locus of the boundary.  $\omega$  is called the *rolling number* of  $M$  because it is the largest radius such that at every boundary point the tangent disk of that radius remains inside  $M$ .

**Theorem 3.12. [Puckering Theorem].** *Let  $M \in \mathbf{R}^2$  be an embedded disk with boundary of class  $\mathcal{K}$ . Then the rolling number (3.13) satisfies*

$$(3.14) \quad \omega(M) \leq \frac{L - \sqrt{L^2 - 4\pi A}}{2\pi}.$$

*If in addition,  $L > 2A$  then*

$$(3.15) \quad \omega(M) \leq \frac{A - \pi - 2\sqrt{3}}{L - 2\pi - 4\sqrt{3}}.$$

*Proof.* Consider the interior parallel domain  $M_a = \{P \in M : \text{dist}(p, \partial M) \geq a\}$  for constant  $a \geq 0$ . If  $a < \omega(M)$  then  $\partial M_a$  is  $C^{1,1}$  embedded circle whose curvature has  $\|\tilde{\kappa}\|_\infty \leq 1/(1-a)$ . Let  $c(s)$  denote an arclength parameterization of  $\partial M$ . Then the tubular neighborhood  $M - M_a$  has a parameterization  $\partial M \times [0, a] \ni (s, t) \mapsto \eta_{c(s)}(t)$  with area form  $(1 - \kappa(s)t)dt ds$ . Estimating the area for  $0 < a < \omega$ ,

$$\begin{aligned} A &\geq \text{area}(M - M_a) \\ &= \int_{\partial M} \int_0^a (1 - \kappa(r)t) dt ds \\ &= aL - \pi a^2. \end{aligned}$$

This holds for small  $a$  so the smaller root provides (3.14).

By the assumption and Proposition 2.1,  $A \geq \pi$  we have  $L > A + \pi$  thus, by (3.14),  $\omega(M) < 1$ . Let  $Z \in \mathcal{C}$  be a point where the distance to  $\partial M$  is a minimum. We argue that  $\mathcal{Y} = \overline{B_{\omega(M)}(Z)} \cap \partial M$  is a doubleton  $\mathcal{Y} = \{Y_1, Y_2\}$  using the structure theorem for the cut locus near the a nonfocal point  $Z$ . Let  $U \subset M$  be a small enough disk about  $Z$ . If  $\#\mathcal{Y} \geq 3$  then one of the components of  $U - \mathcal{C}$ , say corresponding to  $Y_1$ , has a vertex angle  $< \pi$  at  $Z$ . Hence the distance to the cut set as a function on  $\partial M$  near  $Y_1$  cannot have a local minimum at  $Y_1$ . We claim that there must be at least two focal disks in  $M$  whose centers  $Z_1, Z_2 \in \mathcal{C}$  are at least  $\tilde{r} = \sqrt{3 - 2\omega - \omega^2}$  apart, measured along  $\mathcal{C}$ . The existence follows as in Theorem 3.7. The distance estimate follows from Proposition 1.1. Consider the exterior tangent unit disks to  $\partial M$  at  $X_1, X_2$ . If  $\text{dist}_{\mathcal{C}}(Z_1, Z_2) < \sqrt{3}$  then unit disks centered at  $Z_1, Z_2$  must be on opposite sides and disjoint from the tangent disks. Thus by elementary geometry, they must be  $\tilde{r}$  apart.

We apply a scaled version of the proof Lemma 3.6 to estimate the area of  $M_a$ . If the curvature of  $\partial M_a$  satisfies  $\|\tilde{\kappa}\|_\infty \leq (1-a)^{-1}$  and if two  $1-a$  disks centered  $Z_1, Z_2 \in \tilde{\mathcal{C}} = \mathcal{C}$  are at least  $2\sqrt{3}(1-a)$  apart along  $\tilde{\mathcal{C}}$ , the cut set of  $M_a$ , then  $M_a$  has

remote ends so  $\text{area}(M_a) \geq (1-a)^2[\pi + 2\sqrt{3}]$ . One should check that  $\sqrt{3}(1-a) \leq \tilde{r}$ , but this holds for all  $a < \omega < 1$ . The total area can now be estimated

$$\begin{aligned} A &= \text{area}(M - M_a) + \text{area}(M_a) \\ &= \int_{\partial M} \int_0^a (1 - \kappa(s)t) dt ds + \text{area}(M_a) \\ &\geq La - \pi a^2 + (1-a)^2 A_1 \end{aligned}$$

where  $A_1 = \pi + 2\sqrt{3}$ . Hence

$$4\sqrt{3}a \leq \left( (L - 2\aleph)^2 + 8\sqrt{3}(A - A_1) \right)^{\frac{1}{2}} - (L - 2A_1).$$

In view of Corollary 3.8,  $L > 2A$  implies  $A > \pi + 2\sqrt{3}$  hence (3.15) follows.  $\square$

#### 4. A sharp reverse isoperimetric inequality and the extremal figure

In this section we consider the best reverse isoperimetric inequality and the figure which extremizes this inequality. Let  $\mathcal{M}(A)$  denote the space of all embedded closed disks  $M \subset \mathbf{R}^2$  whose boundary curves are in class  $\mathcal{K}$  and whose areas satisfy  $\text{area}(M) = A$ . Let  $\mathcal{N}(L)$  denote the space of all embedded closed disks  $M \subset \mathbf{R}^2$  whose boundary curves are in class  $\mathcal{K}$  and whose length  $\text{length}(\partial M) = L$ . Then we say  $E \in \mathcal{M}(A)$  is extremal if  $\text{length}(\partial E) = \sup\{\text{length}(\partial M) : M \in \mathcal{M}(A)\}$ . Similarly,  $E \in \mathcal{N}(L)$  is extremal if  $\text{area}(M) = \inf\{\text{area}(M) : M \in \mathcal{N}(L)\}$ . Although these problems are dual, they require slightly different treatment (*e.g.* see Lemma 4.16).

**Theorem 4.1.** *The set of pairs  $(A, L)$  where  $A$  is the area and  $L$  is the boundary length of  $M \subset \mathbf{R}^2$ , an embedded closed disk whose boundary is of class  $\mathcal{K}$ , consists exactly of the points in the first quadrant (shown in Figure 4.) satisfying three inequalities:*

- (1) *The isoperimetric inequality*

$$4\pi A \leq L^2.$$

*Equality holds if and only if  $M$  is a circular disk.*

- (2) *The reverse isoperimetric inequality. If  $2\pi \leq L < 14\pi/3$  then there holds*

$$(4.1) \quad \sin\left(\frac{L - 2\pi}{4}\right) \leq \frac{A - \pi}{4}.$$

*Equality holds in (4.1) if and only if  $M$  is congruent to the peanut  $P_\delta$  (Figure 1.) where*

$$\delta = 4 \sin\left(\frac{L - 2\pi}{8}\right).$$



(3) *Embeddedness border.* If  $L \geq 14\pi/3$  then

$$A > \pi + 2\sqrt{3}.$$

Equality cannot hold, although there are arbitrarily nearby regions for which the embeddedness degenerates by “puckering”. For example one can consider a sequence of domains decreasing to the dumbbell region consisting of two unit disks, two triangles with circular sides and a segment of length  $L/2 - 7\pi/3$ .

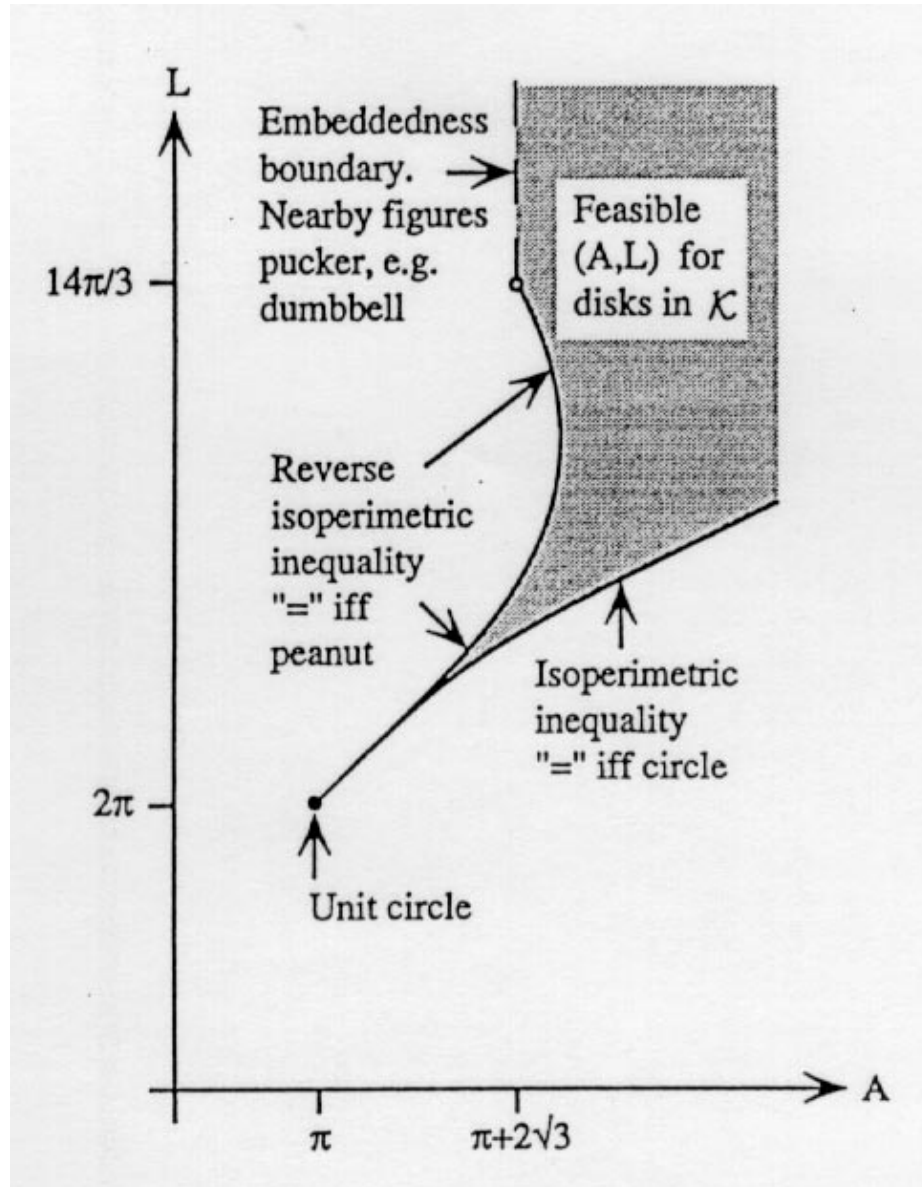


Fig. 4 Attainable  $(A, L)$  for  $M \in \mathcal{K}$ .

First we show that extremal figures exist.

**Theorem 4.2.** *For any  $\pi \leq A \leq \pi + 2\sqrt{3}$  there exists an  $E \in \mathcal{M}(A)$  which is extremal, namely,  $\text{length}(E) = \sup\{\text{length}(\partial M) : M \in \mathcal{M}(A)\}$ .*

*Proof.* For area fixed, consider a maximizing sequence of embedded disks  $M_i \subset \mathcal{M}(A)$  such that  $\text{length}(\partial M_i) \nearrow \ell(A) = \sup\{\text{length}(\partial M) : M \in \mathcal{M}(A)\}$ . Corollary 3.8 shows that if  $M_i \in \mathcal{M}(A)$  then it contains a disk of radius 1 and that  $M_i$  is star-shaped with respect to the center of this disk. Translating if necessary, we assume that these centers lie on the origin. Corollary 3.8 also shows that the lengths are uniformly bounded  $\ell(A) \leq 2A$ . By scaling the parameters, the boundary curves  $\sigma_i : \mathbf{S}^1 \rightarrow \mathbf{R}^2$  are uniformly bounded in  $C^{1,1}(\mathbf{S}^1)$ . It follows by Arzela's Theorem that a subsequence  $\sigma_{i'} \rightarrow \sigma$  uniformly in  $C^1$ , hence with the same bound on  $\mathbf{Lip}(\sigma')$ . Thus  $\sigma \in \mathcal{K}$ . The limit  $M$  must also contain  $B_1(0)$ . Since  $(\rho_i)_s$  are bounded as in Corollary 3.8,  $\partial M_{i'}$  are uniformly transverse to the rays from the origin, except possibly if  $|M| = \pi + 2\sqrt{3}$  and then only for one radius, the same must be true of the limit. Hence we can conclude that  $M$  is topologically an embedded closed disk.  $\square$

**Theorem 4.3.** *For any  $2\pi \leq L < 14\pi/3$  there exists an  $E \in \mathcal{N}(L)$  which is extremal, namely,  $\text{area}(E) = \inf\{\text{area}(M) : M \in \mathcal{N}(L)\}$ .*

*Proof.* Again, since length is bounded, after translating if necessary, the same compactness property shows there exists limits of subsequences. It remains to argue that the limiting figures are embedded. If not, there are elements of the approximating minimizing sequence, call them  $M$  which are nearly degenerate having a thin waist. Let  $\gamma : [0, L] \rightarrow \partial M$  denote an arclength parameterization of the boundary curve. There are points on opposite sides of the waist,  $s_1 + 2\pi < s_2 < s_1 + L - 2\pi$ , so that  $d = \text{dist}(\gamma(s_1), \gamma(s_2))$  is minimum among such points and arbitrarily small. Hence  $\gamma(s_1)$  and  $\gamma(s_2)$  have a common normal line. But the total length must exceed the double of the minimal length for a segment of class  $\mathcal{K}$  connecting the line elements  $\gamma(s_1)$  and  $\gamma(s_2)$ . By Dubins Theorem [D] this segment consists of the three unit arcs forming a lightbulb shape with length  $\pi + 4 \cos^{-1}(1/2 + d/4)$  which exceeds  $L/2$  for  $d$  sufficiently small.  $\square$

First we shall show that extremal figures consist of finitely many unit circular arcs. From the control theory standpoint, viewing the curvature  $\kappa$  as a control, and maximizing the length among curves with curvature  $\kappa$  which are closed, embedded, of class  $\mathcal{K}$ , enclosing an area  $A$ , this shows the *bang-bang* type result that  $\kappa = \pm 1$ . We shall examine variations of short arcs on the boundary of minimizers. First we give our parameterization of pairs of line elements on the endpoints of a short arc using the shortest path.

**Lemma 4.4.** *Suppose  $\sigma$  is an arc of class  $\mathcal{K}$  and  $\ell = \text{length}(\sigma) \leq \pi/2$ . Then the shortest arc  $\lambda$  of class  $\mathcal{K}$  connecting the starting and ending line elements of  $\sigma$  consists of a unit circular arc of angle (integral curvature)  $\alpha$  followed by a line segment of length  $\beta \geq 0$  followed by a unit circular arc of angle  $\gamma$  such that the total length of  $\lambda$  is  $|\alpha| + \beta + |\gamma| \leq \pi/2$ . ( $\alpha < 0$  corresponds to a concave arc.)*

*Proof.* Dubins [D] showed that the minimizer consists of either three unit circular arcs or arc-segment-arc. By Dubins' Proposition 6, the osculating disks on the opposite side of  $\gamma$  at  $\gamma(0)$  and  $\gamma(\ell)$  are either tangent, in which case the shortest connecting arc of class  $\mathcal{K}$  consists of a subarc of the first circle followed by a subarc of the second; or they are disjoint. By Proposition 1.1 neither osculating disk at one endpoint contains the other endpoint in the interior. Hence, among all straight line segments and circular arcs tangent to an osculating disk at one endpoint and to another at the other endpoint, the straight line segment between closest disks gives the shortest curve between elements.  $\square$

Our local considerations will examine which boundary subarcs of  $M \in \mathcal{M}(A)$  can be lengthened. To do this, we describe various pairs of endpoint line elements. Let  $\sigma \subset \partial M$  be a subarc of length  $\ell = \text{length}(\sigma) \leq \pi/3$  oriented in the usual direction of  $\partial M$ . Let  $(\alpha, \beta, \gamma)$  parameterize the shortest arc of class  $\mathcal{K}$  connecting endpoint elements of  $\sigma$ . If  $\alpha \geq 0$  and  $\gamma \geq 0$  we say that  $(\alpha, \beta, \gamma)$  is a *convex* pair of line elements. If  $\alpha \leq 0$  and  $\gamma \leq 0$  we say *concave* and if neither we say *mixed*.

**Lemma 4.5.** *Given  $0 < l \leq \pi/3$ , let  $\sigma : [0, l] \rightarrow \mathbf{R}^2$  be a unit speed arc of class  $\mathcal{K}$  and let  $\{\mathbf{t}(t), \mathbf{n}(t)\}$  be an orthonormal frame at  $\sigma(t)$  so that  $\sigma' = \mathbf{t}$ . Let  $D^+(t)$  and  $D^-(t)$  be the closed osculating unit disks to  $\sigma$  at  $\sigma(t)$  on the  $\pm \mathbf{n}$  sides of  $\sigma$ . Then*

- (1)  $\sigma$  is an embedded arc;
- (2) *Disks on the same side always intersect:  $D^+(0) \cap D^+(l) \neq \emptyset$  and  $D^-(0) \cap D^-(l) \neq \emptyset$ ;*
- (3) *Disks on the opposite sides don't intersect: For  $0 \leq l \leq \pi/2$ , either  $D^-(0) \cap D^+(l) = \emptyset$  or  $D^-(0)$  is tangent to  $D^+(l)$  and  $\lambda$  is the class  $\mathcal{K}$  curve consisting of an arc of  $\partial D^-(0)$  followed by an arc of  $\partial D^+(l)$ .*

*Proof.* The embeddedness follows from the Schur-Schmidt Proposition 1.1. That intersection occurs is a geometric exercise. For example, by moving  $D^+(0)$  and  $D^+(l)$  as far apart as possible, we may assume that  $\sigma$  is the shortest arc  $\lambda$  in  $\mathcal{K}$  between the line elements  $(\sigma(0), \sigma'(0))$  and  $(\sigma(l), \sigma'(l))$ . By Lemma 4.4 we may suppose  $\lambda$  consists of at most three pieces consisting of (a subset) of a unit circular subarc of angle  $\alpha$ , a line segment of length  $\beta \geq 0$ , and another unit circular arc of angle  $\gamma$ . If, *e.g.*,  $\alpha \geq 0$  and turning so the segment is in the  $\partial/\partial x$  direction, we see that the centers of  $D^+(0)$  and  $D^+(l)$  are at most

$$d^2 = (\Delta x)^2 + (\Delta y)^2 = (2 \sin \alpha + \beta + 2 \sin \gamma)^2 + (2 \cos \alpha - 2 \cos \gamma)^2$$

apart, where  $\alpha + \beta + \gamma \leq \pi/3$  and  $\gamma \geq 0$ . Because, *e.g.*,  $\cos(\alpha) \geq 1/2$ , by examining the gradient, we see that the maximum value occurs when  $\beta = 0$  making  $d = 2$ . In other words,  $D^+(0) \cap D^+(l) \neq \emptyset$ . Similarly for  $D^-$ . (3) is Dubins' Proposition 6, [D].  $\square$

Thus, by Proposition 1.1, the segment  $\sigma \in \mathcal{K}$  of length  $\text{length}(\sigma) \leq \pi/3$  must lie in the region  $N$  determined by the endpoints of  $\sigma$  and their directions, where  $N$  is the bounded component of the complement of  $D^+(0) \cup D^-(0) \cup D^+(l) \cup D^-(l)$ .

**Lemma 4.6.** *In the notation of Lemma 4.5, there are unit circles which are tangent to both  $D^+(0)$  and  $D^+(l)$ . Let  $\mu^+$  be the short subarc between the points of tangency which touches  $N$ ; similarly define  $\mu^-$ . Then  $\mu^+, \mu^- \subset \bar{N}$ .*

*Proof.* Notice that by Lemma 4.5 (3), either  $\sigma$  consists of two circular segments, in which case  $\mu^+$  coincides with  $\sigma$ , or  $D^-(0) \cap D^+(l) = \emptyset$  and  $D^-(l) \cap D^+(0) = \emptyset$ . Then a unit disk rolled from  $D^-(0)$  along  $D^+(0)$  bumps into  $D^+(l)$  on the “inside” of  $N$ .  $\square$

Denote the “lips shaped” subregion of  $\bar{N}$  between and including  $\mu^+$  and  $\mu^-$  by  $\mathcal{L}$  as shown in Figure 5. Denote the boundary arc from  $\sigma(0)$  to  $\sigma(l)$  consisting of concatenations of the subarcs of  $\partial D^+(0)$ ,  $\mu^+$  and  $\partial D^+(l)$  by  $\lambda^+$ . Define  $\lambda^-$  analogously. Then  $\lambda^\pm \in \mathcal{K}$  have the same starting and ending line elements as  $\sigma$ . Let  $\lambda$  be the shortest curve of class  $\mathcal{K}$  between the starting and ending line elements of  $\sigma$  with parameters  $(\alpha, \beta, \gamma)$ . If  $l \leq \pi/3$  then by Lemma 4.4,  $\lambda$  consists of a unit circle arc of angle  $\alpha$  followed by a straight segment of length  $0 \leq \beta$  followed by an arc of angle  $\gamma$  so that (up to reflection)  $|\gamma| \leq |\alpha|$  and  $|\alpha| + \beta + |\gamma| \leq \pi/3$ . We call  $(\alpha, \beta, \gamma)$  the *parameters* of  $\lambda$ .

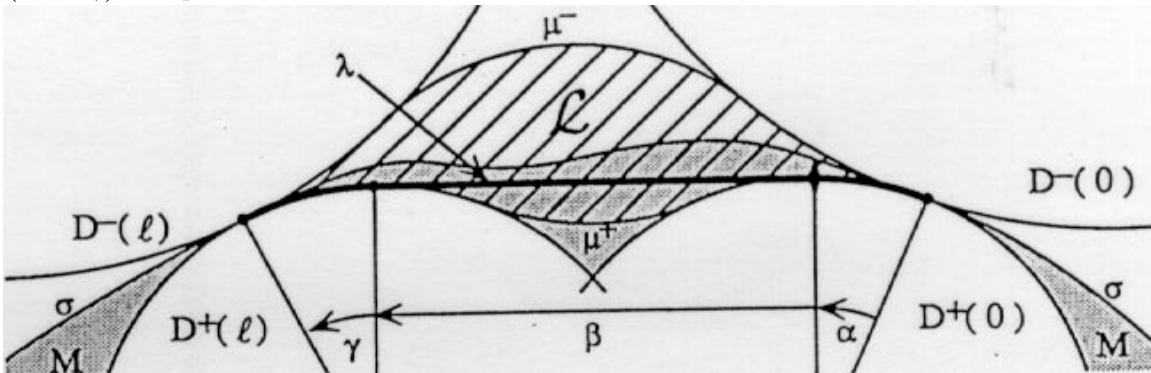


Fig. 5 “Lips” domain.

**Lemma 4.7.** *Let  $\sigma : [0, l] \rightarrow \mathbf{R}^2$  be a segment of class  $\mathcal{K}$ . We suppose that the osculating disks  $D^+(0) \cap D^+(l) \neq \emptyset$  and  $D^-(0) \cap D^-(l) \neq \emptyset$ , for example if  $0 < l \leq \pi/3$ . Let  $\mathcal{L}$  be the lips-shaped region determined by the starting and ending line elements of  $\sigma$ , and  $\lambda^\pm$  the corresponding boundary arcs of  $\mathcal{L}$ . First  $\sigma \subset \mathcal{L}$ . Moreover, then either*

- (1)  $\sigma$  consists of one or two circular arcs in which case  $\lambda^+ = \lambda = \lambda^-$ ; This is the case if and only if the curvature is a.e. piecewise equal to  $\pm 1$  with at most one essential sign change.
- (2)  $\sigma = \lambda^+$ , ( $\sigma = \lambda^-$ , resp.) in which case the curvature is a.e.  $\kappa = 1$  for  $0 \leq t < l_1^+$ ,  $\kappa = -1$  for  $l_1 \leq t < l_2^+$  and  $\kappa = 1$  for  $l_2^+ \leq t \leq l$  (or its negative, resp.) where  $l_1^+$  is the length of the first arc and  $l_2^+$  the length of the second. Note that  $l_i^\pm$  are determined by the starting and ending line elements of  $\lambda$ .
- (3)  $\lambda^+ \not\equiv \sigma \not\equiv \lambda^-$  which is the case if  $\kappa$  is any other function. Let  $\lambda$  be the

shortest curve of class  $\mathcal{K}$  between starting and ending line elements of  $\sigma$  with parameters  $(\alpha, \beta, \gamma)$ . Suppose  $\alpha, \gamma \geq 0$  then  $\text{length}(\sigma) < \text{length}(\lambda^-)$ .

*Proof.* Case (1) occurs if the osculating circles  $D^+(0)$  and  $D^-(l)$  ( or  $D^-(0)$  and  $D^+(0)$ ) are tangent and Proposition 1.1 (3) applies and  $\sigma \in \bar{m}L$ . Suppose this is not the case for the rest of the proof. By constructing a field of semicircles by translating  $\mu^\pm$  inside  $N - \mathcal{L}$  one sees by the maximum principle that  $\lambda \subset \bar{\mathcal{L}}$ .

(3) will be demonstrated in two steps. First we claim that  $\lambda^-$  and  $\sigma$  have a common perpendicular. To see this, orient  $\sigma$  so that  $\lambda(|\alpha|) = 0$  and  $\lambda'(|\alpha|) = \partial/\partial x$ . By the intermediate value theorem there is a point on  $\sigma$  so that  $\sigma'(t_0) = \partial/\partial x$ . Put  $\sigma'(s) = \exp(i\vartheta(s))$ ; similarly define  $\vartheta^\pm$ . Because  $\|\vartheta(t) - \vartheta(t_0)\| \leq |t - t_0| \leq \pi/3$  we have that  $\sigma$  is a graph over the  $x$ -axis. Similarly, so are  $\lambda^\pm$ . Let  $a < b$  be the  $x$ -coordinates of the endpoints of  $\sigma$  and let  $a < x_1 < x_2 < b$  be the  $x$ -coordinates of the first and second jumps of the curvature  $\kappa^-$  of  $\lambda^-$  (so they are the endpoints of of  $\mu^-$ ). Observe that  $\vartheta(a) = \vartheta^\pm(a)$  and

$$\vartheta^-(x_1) \leq \vartheta(a) + \int_a^{x_1} \kappa ds = \vartheta(x_1).$$

Also  $\vartheta^-(x_2) \geq \vartheta(x_2)$ . Because  $\mu^-$  is a circular arc, for each  $\vartheta^-(x_1) \leq \zeta \leq \vartheta^-(x_2)$  there is a unique  $x$ -coordinate  $\xi(\zeta)$  where the tangent of  $\mu^-$  is in the  $\zeta$  direction  $(\mu^-)'(\xi(\zeta)) = \exp(i\zeta)$ . Consider the continuous function  $f(x) = \langle \sigma'(x), \mu^-(\xi(\vartheta(x))) - \sigma(x) \rangle$  which measures the distance between the normal lines thru  $\sigma(x)$  and  $\mu^-$  at the corresponding angles. Observe that  $f(x_1) \geq 0$  and  $f(x_2) \leq 0$ . By the intermediate value theorem, there is an  $x_3 \in [x_1, x_2]$  where the normal lines of  $\sigma$  and  $\mu^+$  coincide. Hence by a rotation, we may assume that this line is the  $y$ -axis and that  $\sigma'(0) = (\mu^-)'(0) = \partial/\partial x$ .

Next we show that  $\text{length}(\mu^-) > \text{length}(\sigma)$ . It suffices to compare the lengths of the parts when  $x \geq 0$  and  $x \leq 0$  separately. Let  $a \geq 0$  be the end  $x$ -coordinate of  $\lambda$  and  $0 \leq x_1 \leq a$  the coordinate of the jump in  $\kappa^-$ . So  $x_1 < 1$ . We have

$$(4.2) \quad \frac{d\vartheta}{dx} = \frac{d\vartheta}{ds} \frac{ds}{dx} = \kappa \sec \vartheta$$

$$\frac{d\vartheta^-}{dx} = \frac{d\vartheta^-}{ds^-} \frac{ds^-}{dx} = \sec \vartheta \begin{cases} +1, & \text{if } x < x_1, \\ -1, & \text{if } x_1 < x < a. \end{cases}$$

Comparison of solutions using  $\vartheta(0) = \vartheta^-(0)$  on the interval  $[0, x]$  for  $x \leq x_1$  and using  $\vartheta^-(a) = \vartheta(a)$  on the interval  $[x, a]$  shows that  $|\sin \vartheta(x)| \leq \sin \vartheta^-(x)$  for all  $0 \leq x \leq a$ . Hence

$$\frac{ds^-}{dx} = \sec \vartheta^-(x) \geq \sec \vartheta(x) = \frac{ds}{dx}$$

on  $[0, a]$  so  $\text{length}(\mu^-) > \text{length}(\sigma)$  follows. □

The following is a technical result that says that any convex arc of class  $\mathcal{K}$  which is not a subcircle spanning convex line elements also contains noncircular subarcs of arbitrarily small size.

**Lemma 4.8.** *Let  $\sigma \subset \mathbf{R}^2$  be a convex arc of class  $\mathcal{K}$  and length  $\ell = \text{length}(\sigma) \leq \pi/3$ . Let  $\lambda$  be the shortest arc of class  $\mathcal{K}$  which spans the endpoint line elements of  $\sigma$ , and let  $(\alpha, \beta, \gamma)$  be the parameters of  $\lambda$ . Suppose  $\beta > 0$ . Then there are positive constants  $c_7, c_8$  depending only on  $\beta$  and  $\ell$  so that if  $\rho < c_7$  then there is a subinterval  $\sigma' \subset \sigma$  with  $\text{length}(\sigma') = \rho$ , and such that  $\beta' \geq c_8\rho$  where  $(\alpha', \beta', \gamma')$  are the parameters of the shortest arc  $\lambda' \in \mathcal{K}$  spanning the endpoint elements of  $\sigma'$ .*

*Proof.* The idea is that by a measure theoretic argument, there is a short subarc  $\sigma'$  that has relatively little total curvature. Hence by computing the worst possible case, namely for the longest convex arc  $\nu$  of class  $\mathcal{K}$  spanning the end elements of the subarc, we can estimate what fraction the  $\beta'$  parameter is of the length of  $\nu$ .

By convexity and continuity, there is a  $0 < \delta < \beta/2 \leq \ell/2$  so that if  $x$  satisfies  $\ell - \delta \leq x \leq \ell$ , then the parameters of  $\sigma|[0, x]$  satisfy  $\alpha_x \geq 0$ ,  $\gamma_x \geq 0$  and  $\beta_x \geq \beta/2$ . Thus for  $c_7$  sufficiently small and any  $\rho < c_7$ , there is an integer  $j = j(\rho) \in \mathbf{N}$  so that  $\ell - \delta < j\rho \leq \ell$ . The total curvature of one of the subintervals of length  $\rho$  must be less than the average curvature, hence for some  $1 \leq m \leq j$ ,

$$(4.3) \quad K' = \int_{[\rho(m-1), \rho m]} \kappa ds \leq \frac{\alpha + \gamma}{j} \leq \frac{\ell - \beta}{j} \leq \frac{\ell - \beta}{\ell - \frac{1}{2}\beta} \rho.$$

Let  $\sigma'$  be this subarc and  $\lambda'$  the shortest arc of class  $\mathcal{K}$  with the same end elements. Now orient  $\lambda'$  so  $\beta'$  is in the  $-\partial/\partial x$  direction (abusing notation). On the other hand, by convexity of  $\sigma'$ , the longest possible convex span,  $\nu \in \mathcal{K}$ , with the same end elements as  $\sigma'$ , would be the curve consisting of a straight line segment  $\nu_1$  of slope  $-\tan \alpha'$ , a unit circular arc  $\nu_2$  of angle  $\alpha' + \gamma'$  and a straight line segment  $\nu_3$  of slope  $\tan \gamma'$ . We now wish to estimate a lower bound for the length of  $\beta'$  in terms of the length of  $\nu$ . Because the maximum  $y$ -coordinate of  $\nu$  occurs at  $\alpha'$  along the circular part of  $\nu$ , the translates of  $\nu_1$ ,  $\nu_3$  and  $\beta$  form a triangle. Hence, the total horizontal component of the straight segments of  $\nu$  is  $\beta$ . Thus, solving for the  $x$  components of the segments  $\nu_1$  and  $\nu_3$ , we obtain  $\nu_1$  runs a horizontal distance  $\beta' \tan \gamma' / (\tan \alpha' + \tan \gamma')$  and  $\nu_3$  for a horizontal distance  $\beta' \tan \alpha' / (\tan \alpha' + \tan \gamma')$ . Hence  $\text{length}(\nu) - \text{length}(\lambda') = \beta'(\sin \alpha' + \sin \gamma') / \sin(\alpha' + \gamma') \leq \rho / \sin(\rho)$ , because  $\alpha' + \beta' \leq \rho$ . Therefore  $\text{length}(\nu) / \text{length}(\lambda') \leq \rho / \sin \rho$ . Hence, using (4.3),

$$\beta' \geq \text{length}(\lambda') - K' \geq \left( \frac{\sin \rho}{\rho} - \frac{\ell - \beta}{\ell - \frac{1}{2}\beta} \right) \rho.$$

Thus for  $c_7$  sufficiently small and  $\rho < c_7$ , the lemma follows.  $\square$

To construct deformations, we first estimate an outward bump on a convex but not entirely unit circular arc.

**Lemma 4.9.** *Let  $\sigma \subset \mathbf{R}^2$  be a convex arc of class  $\mathcal{K}$  and length  $\ell = \text{length}(\sigma) \leq \pi/3$ . Let  $\lambda$  be the shortest arc of class  $\mathcal{K}$  which spans the endpoint line elements of  $\sigma$ , and let  $(\alpha, \beta, \gamma)$  be the parameters of  $\lambda$ . Assume that  $\beta \geq k\ell$  for some constant  $k > 0$ . Let  $\mathcal{L}$  be the lips shaped region determined by the elements of  $\sigma$  and  $\lambda^-$  the “outer” boundary portion of  $\mathcal{L}$  spanning the elements of  $\sigma$ . Then there are positive constants  $c_1, c_2, c_3$  depending only on  $k$  such that*

$$\begin{aligned} \text{length}(\lambda^-) - \text{length}(\sigma) &\geq c_1 \ell^3, \\ \text{area enclosed by}(\lambda^- - \sigma) &\leq c_2 \ell^3 \end{aligned}$$

whenever  $\ell \leq c_3$ .

*Proof.* By convexity, the largest possible convex arc  $\nu \in \mathcal{K}$  consists of linear extensions of the endpoint line elements and a circular fillet as in Lemma 4.8. By symmetry, we may suppose  $\alpha \leq \gamma$ . Suppose  $\lambda$  is positioned so that  $\beta$  is in the  $-\partial/\partial x$  direction. Assume that the center of the  $\alpha$  circle is the origin  $O$ . Then the center of the  $\gamma$  circle is  $P = (-\beta, 0)$ . The center of the fillet is  $Q = (-\beta \tan \gamma / (\tan \alpha + \tan \gamma), \beta \sin \alpha \sin \gamma / (\sin \alpha + \sin \beta))$ . Thus  $|\nu| = |OQ| + \alpha + \gamma + |QP|$  where  $|OQ| = \beta \sin \gamma / (\sin \alpha + \sin \gamma)$ .  $\lambda^-$  consists of three circular arcs. The center  $R$  of the central circular arc is an intersection point of two circles of radius 2 with centers at  $(2 \sin \alpha, 2 \cos \alpha)$  and  $(-\beta - 2 \sin \gamma, 2 \cos \gamma)$ . Thus  $R$  is above the line  $[O, Q]$  by a distance we can estimate in terms of  $\text{length}([O, Q])$ . Now use the notation  $f \sim g$  to mean there is a positive constant  $c$  depending only on  $k$  (and not  $\alpha + \gamma$ ) such that  $c^{-1}f \leq g \leq cf$ . Then we see (using  $\alpha + \gamma \leq \pi/3$  and  $\gamma \geq \alpha$ ) that  $|OQ| \sim \beta^2$ . Using the circle of radius 2,  $|QR| \sim \beta^2$ . But this means that  $\lambda^+$  is a displacement of the tangent line of  $\nu(0)$  by an amount  $|QR|$  over a distance  $\sim \beta$ . It follows that  $|\lambda^+| - |\nu| \sim \beta^3$ . Also the area increase  $\text{area}(\lambda^- - \nu) \leq \Delta A \leq \text{area}(\lambda^- - \lambda)$ . However, because the  $y$ -coordinate of  $Q$  is  $\leq \beta^2$  we have  $\Delta A \sim |\lambda| \beta^2 \sim \beta^3$  because  $\lambda \sim \beta$  by assumption.  $\square$

Also we describe the inward deformation which is constructed by displacing inward part of a convex noncircular arc and connecting.

**Lemma 4.10.** *Let  $\sigma \subset \mathbf{R}^2$  be a convex arc of class  $\mathcal{K}$  and length  $\ell = \text{length}(\sigma) \leq \pi/3$  and let  $\lambda$  be the corresponding shortest arc of class  $\mathcal{K}$  spanning the endpoint elements of  $\lambda$  and assume the parameter  $\beta > 0$ . Assume that  $\sigma$  is oriented so that  $\beta$  is in the  $\partial/\partial x$  direction so that  $\sigma$  and  $\lambda$  graphs over  $a \leq x \leq b$ . Let  $\sigma_1$  and  $\sigma_2$  be two disjoint subarcs of  $\sigma$  corresponding to coordinates  $[a, a + \ell_1]$  and  $[b - \ell_2, b]$ . Let  $\lambda_i$  be the minimal arcs of class  $\mathcal{K}$  which span the endpoint line elements of the  $\sigma_i$ , and let  $(\alpha_i, \beta_i, \gamma_i)$  be the parameters of  $\lambda_i$ . Assume that for some constant  $k > 0$  there holds the inequality  $\beta_i \geq k\ell_i$ . Let  $\mathcal{L}_i$  be the lips shaped regions determined by the elements of  $\sigma_i$  and  $\lambda_i^+$  the “inner” boundary portion of  $\mathcal{L}_i$  spanning the elements of  $\sigma_i$ . Let  $h_i(\ell_i) = \sup\{\lambda_i^+(x) - \sigma(x)\}$  be the maximum height of  $\lambda_i^+$  above  $\sigma$ . Because  $\mathcal{L}_i$  depends continuously on  $\ell_i$ , by continuously decreasing one of the  $\ell_i$ , we*

may arrange that  $\ell_i = \ell_i(l)$  are continuous functions so that  $l = \max\{\ell_1(l), \ell_2(l)\}$  and  $h(l) = h_1(\ell_1(l)) = h_2(\ell_2(l))$ . This means that there are continuous functions  $y_1(l) \in (a, a + \ell_1(l))$  and  $y_2(l) \in (b - \ell_2(l), b)$  so that the function

$$\sigma_l(x) = \begin{cases} \sigma(x), & \text{if } x \leq y_1 \text{ or } y_2 \leq x; \\ \sigma(x) + h(l), & \text{if } y_1(l) < x < y_2(l). \end{cases}$$

is of class  $\mathcal{K}$ . Then there are positive constants  $c_4, c_5, c_6$  depending only on  $k$  such that

$$\begin{aligned} \text{length}(\sigma_l) - \text{length}(\sigma) &\geq -c_4 l^3, \\ \text{area enclosed by } (\sigma - \sigma_l) &\geq c_5 l^2 \end{aligned}$$

whenever  $l \leq c_3$ .

*Proof.* This follows from Lemma 4.9 and its arguments. Using the same notations, the vertical displacement is  $\sim l^2$  so that the area displacement is also  $\sim l^2$ . The length change happens only on the bumps, so it is  $\sim l^3$ .  $\square$

**Lemma 4.11.** *For  $A \leq \pi + 2\sqrt{3}$ , let  $M \subset \mathcal{M}(A)$  be an embedded closed disk. Let  $\sigma \subset \partial M$  so that  $\text{length}(\sigma) \leq \pi/3$ . Let  $\lambda$  be the shortest arc of class  $\mathcal{K}$  connecting the endpoint line elements of  $\sigma$  and  $(\alpha, \beta, \gamma)$  its parameters. Assume that  $(\partial M - \sigma) \cap \mathcal{L} = \emptyset$ . If  $\beta > 0$  and  $\sigma \neq \lambda^\pm$  (the boundary curves of  $\mathcal{L}$  as described in Fig. 5) then there is a deformation  $M_\varepsilon \in \mathcal{M}(A)$ , so that  $\text{length}(\partial M_\varepsilon) > \text{length}(\partial M)$ .*

*Proof.* First we remark that it is sufficient to find a deformation which increases the length and merely doesn't increase the area. Let  $M_\varepsilon$  be such a deformation. By Corollary 3.8 we have shown that  $M_\varepsilon$  is starshaped with respect to some point  $P \in M_\varepsilon$ . By the intermediate value theorem, there is a line through  $P$  so that  $\partial M_\varepsilon$  is parallel at the two intersection points with the line. Cutting along the line and splicing in a parallelogram of suitable width increases the length and adjusts the area to  $A$ . (Alternately we could scale the domain.)

Now we consider various cases of  $\sigma$ . Suppose it is possible to find a subarc  $\sigma_1 \subset \sigma$  so that the line elements determined by  $\sigma_1$  are concave, and  $\sigma_1$  does not coincide with the interior boundary of the lips shaped regions determined by the endpoint elements of  $\sigma_1$ . By Lemma 4.7, replacing  $\sigma_1$  by  $\lambda^+$  increases the length, and since  $\sigma_1 \neq \lambda^+$ , decreases the area.

Thus if  $\lambda$  is concave we are finished. Similarly if  $\lambda$  is nontrivially mixed, say  $\alpha > 0$  and  $\gamma \leq 0$ . Assume  $\beta$  is oriented along  $\partial/\partial x$ . The idea is that by continuity,  $\sigma$  must go from nonpositive to nonnegative. Hence there is a point  $x \in \sigma \cap \beta$  where the direction of  $\sigma$  is increasing. If  $x \neq \beta \cap \gamma$  or  $\gamma \neq 0$  then the interval from  $x$  to  $\sigma(\ell)$  is concave. If  $x = \sigma(\ell)$  then there is an  $x$  interior to  $\sigma$  with  $\sigma(x)$  negative and with tangent line above  $D^-(\ell)$ . Then the interval from  $x$  to  $\ell$  is concave.



The remaining case is  $\lambda$  is convex and  $\sigma$  contains no concave subintervals. In this case  $\sigma$  is convex and we suppose that  $\beta$  is oriented in the  $\partial/\partial x$  direction. Hence  $\sigma$  is a convex function and  $M$  is above the graph. Since  $\beta > 0$  we have

$$\alpha + \gamma = \vartheta(\ell) - \vartheta(0) = \int_{\sigma} \kappa ds < \ell.$$

Choose an integer  $k$  so that  $0 < \ell/k < \ell - \alpha - \beta$ . Hence, dividing  $\sigma$  into  $4k$  equal intervals, there must be four disjoint subintervals of length  $\ell/4k$ , call three of them  $\sigma_1, \sigma_3$  and  $\sigma_4$  so that  $\int_{\sigma_i} \kappa ds < \ell/4k$  and so that  $\text{dist}(\sigma_1, \sigma_3) \geq 1/4k$ . We shall construct an outward bump on  $\sigma_4$  which increases the length and area, and an inward bump stretching from  $\sigma_1$  to  $\sigma_3$  which decreases the area, but also possibly decreases the length. We shall show that together these form the desired perturbation. By Lemma 4.8, there is  $c_7$  sufficiently small, so for every  $\delta < c_7$ , there are subintervals of size  $\delta$  in  $\sigma_i$  with  $\beta > c_8\delta$ . Hence, using Lemma 4.9 we construct on these intervals an outward bump in  $\sigma_4$  so that the change in area and length become

$$\Delta L_4 \geq c_1\delta^3; \quad \Delta A_4 \leq c_2\delta^3.$$

For a different  $l \leq c_7$  we use Lemma 4.9 to construct an inward deformation on the pair of intervals from  $\sigma_1$  and  $\sigma_3$  so that

$$\Delta L_{13} \geq -c_4l^3, \quad \Delta A_{13} \leq -c_5l^2,$$

where the extra interval insures uniformly positive width. Now one simply chooses  $c > 0$  and puts  $l^2 = c\delta^3$  so that  $\Delta A = \Delta A_4 + \Delta A_{13} \leq 0$  and observes that  $\Delta L > 0$  for  $l$  sufficiently small to finish the argument.  $\square$

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