BLASCHKE'S ROLLING THEOREM FOR MANIFOLDS WITH BOUNDARY

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ABSTRACT. For a complete Riemannian manifold M with compact boundary ∂M denote by $\mathcal{C}_{\partial M}$ the cut locus of ∂M in M. The ***rolling radius*** of M is $\operatorname{Roll}(M) := \operatorname{dist}(\partial M, \mathcal{C}_{\partial M})$. (When M is a compact domain in Euclidean space this agrees with the definition given by Blaschke.) Let $\operatorname{Focal}(\partial M)$ be the focal distance of ∂M in M. When M is a strictly convex domain in Euclidean space Blaschke's rolling theorem is the equality $\operatorname{Roll}(M) = \operatorname{Focal}(\partial M)$. In this note we give other conditions that imply $\operatorname{Roll}(M) = \operatorname{Focal}(\partial M)$. In particular Blaschke's theorem holds if: (1) The Ricci tensor Ric of M is non-negative and the mean curvature H of ∂M with respect to the inward normal is positive. (2) The sectional curvature of M is non-negative and at every point of ∂M are least $(\dim M)/2$ of the principal curvatures of ∂M with respect to the inward normal are positive. (3) M is the complement of a bounded star like domain D with Euclidean space. Also in (1) if the condition on the mean curvature is weakened to just being non-negative there is a rigidity result: All counterexamples to Blaschke's theorem are either products $\partial M \times [0, b]$ or "generalized Möbius bands". These results extend to more general curvature conditions.

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1. INTRODUCTION

Let $M \subset \mathbf{R}^n$ be a bounded domain with smooth boundary. The ***rolling radius*** of M (denoted by $\operatorname{Roll}(M)$) is the largest number r so that for each $x \in \partial M$ there is an open ball $B(y, r) \subseteq M$ with $x \in \overline{B}(y, r)$ (the closure of $B(y, r)$). That is it is possible to roll a ball of radius r along ∂M and stay inside of M. If M is strictly convex (that is the principal curvatures $\lambda_1, \ldots, \lambda_{n-1}$ of ∂M with respect to the inward normal are strictly positive) then:

Blaschke's Rolling Theorem [3, 19, 25, 8] If M is strictly convex

$$\operatorname{Roll}(M) = \inf\left\{\frac{1}{\lambda_i(x)} : x \in \partial M, \quad 1 \leq i \leq n-1\right\}.$$

In this note this is extended to more general domains and to Riemannian manifolds with boundary. Let M be a complete Riemannian manifold with nonempty boundary ∂M and $\mathbf{n}$ be the inward pointing unit normal to ∂M. For each $x \in \partial M$ let $\gamma_x(t) := \exp_x(t\mathbf{n}(x))$ be the unit speed geodesic starting at x with $\gamma_x'(0) = \mathbf{n}(x)$. The ***cut point*** of $x \in \partial M$ is the point (if one exists) $\operatorname{Cut}_{\partial M}(x) := \gamma_x(t_0)$ where γ_x stops minimizing the distance to ∂M. The ***cut locus*** of ∂M is

$$\mathcal{C}_{\partial M} := \{\operatorname{Cut}_{\partial M}(x) : x \in \partial M\}.$$

If $x \in \partial M$ and $r = \operatorname{dist}(x, \mathcal{C}_{\partial M})$, then the open ball $B(\operatorname{Cut}_{\partial M}(x), r)$ is disjoint form ∂M, $x \in \overline{B}(\operatorname{Cut}_{\partial M}(x), r)$ and this is not true for any ball of larger radius. Thus the natural generalization of the rolling radius to complete Riemannian manifolds with boundary is

$$\operatorname{Roll}(M) := \operatorname{dist}(\partial M, \mathcal{C}_{\partial M}).$$

If $x \in \partial M$, the ***focal distance*** (which may be infinite) of x is the number $t = t_0$ so that x stops being a local minimum of the function $\operatorname{dist}(\gamma_x(t), \cdot)$ defined on ∂M. Let $\operatorname{focal}_{\partial M}(x)$ be the focal distance of $x \in \partial M$ and set

$$\operatorname{Focal}(\partial M) := \min\{\operatorname{focal}_{\partial M}(x) : x \in \partial M\}.$$

Clearly $\operatorname{Roll}(M) \leq \operatorname{Focal}(M)$. If M is a strictly convex domain in $\mathbf{R}^n$ then by standard results $\operatorname{Focal}(M) = \inf\{1/\lambda_i(x) : x \in \partial M, \ \lambda_i(x) > 0, \ 1 \leq i \leq n-1\}$ so if M is strictly convex then Blaschke's rolling theorem is the equality $\operatorname{Roll}(M) = \operatorname{Focal}(\partial M)$.

For general Riemannian manifolds with boundary it is easier to compute, or at least estimate, $\operatorname{Focal}(\partial M)$ in terms of curvature properties of M and ∂M than it is to estimate $\operatorname{Roll}(M)$ directly. Thus it is also interesting in the more general case of Riemannian manifolds with boundary to know when $\operatorname{Roll}(M) = \operatorname{Focal}(\partial M)$.

Theorem 1. *Let M be a complete Riemannian manifold with compact boundary so that the Ricci tensor satisfies* $\operatorname{Ric} \geq 0$ *and the mean curvature of ∂M with respect to the inward normal is positive. Then* $\operatorname{Roll}(M) = \operatorname{Focal}(\partial M)$.

Theorem 2. *Let M be a complete Riemannian manifold of dimension n with compact boundary so that the sectional curvature of M satisfies $K_M \geq 0$ and at each point $x \in \partial M$ at least $n/2$ of the principal curvatures of ∂M with respect to the inward normal are positive. Then* $\operatorname{Roll}(M) = \operatorname{Focal}(\partial M)$.

Generalizations to the case of arbitrary lower bounds on the Ricci or sectional curvature are given in Theorems 2.3 and 2.4 below. The proofs are based on a lemma, motivated by results of Klingenberg [17, 18], which states that $\operatorname{Roll}(M) < \operatorname{Focal}(\partial M)$ implies the existence of a geodesic segment in M

perpendicular to ∂M at its endpoints, with its midpoint in $\mathcal{C}_{\partial M}$ and whose length is a local minimum for all curves with both end points on ∂M.

There is a rigid version of Theorem 1 under the assumption the mean curvature is only nonnegative. A ***Riemannian cylinder*** is a Riemannian product $N \times [0, b]$ for some compact Riemannian manifold N and some $b > 0$. Let N be a compact Riemannian manifold with a fixed point free isometry $\theta \colon N \to N$ of order two (that is $\theta \circ \theta = \mathrm{Id}$). Then the ***generalized Möbius band*** based on N and θ is $N \times [0, b]/\sim$ where $\sim$ is the equivalence relation that identifies (x, b) with $(\theta(x), b)$. If the Ricci of N is non-negative, then both the Riemannian cylinder and the generalized Möbius band have non-negative Ricci tensor, totally geodesic boundary, and $\mathrm{Roll}(M) < \mathrm{Focal}(\partial M) = \infty$. These examples show that in Theorem 1 it is not possible to replace "positive mean curvature" with "non-negative mean curvature". However these are the only counterexamples:

Theorem 3. *Let M be a complete connected Riemannian manifold with smooth non-empty compact boundary ∂M so that the Ricci tensor of M satisfies* $\mathrm{Ric} \geq 0$ *and with the mean curvature H of ∂M with respect to the inward normal is non-negative. If* $\mathrm{Focal}(\partial M) > \mathrm{Roll}(M)$ *then M is either a Riemannian cylinder or a generalized Möbius band.*

This is closely related to a rigidity theorem of Galloway [12] and the warped product splitting theorems of Kasue [16] and Croke-Kleiner [7]. If M is a domain in Euclidean space then it is never a Riemannian cylinder or generalized Möbius band so this extends Blaschke's theorem $\mathrm{Roll}(M) = \mathrm{Focal}(\partial M)$ from the class of convex domains to the class of "mean convex" domains (domains with non-negative mean curvature). This allows the topology of the domains to be more complicated, as every convex domain is diffeomorphic to an open ball, but mean convex sets can be much more complicated. For example every compact oriented surface can be realized as the boundary of a domain in $\mathbf{R}^3$ with positive mean curvature. Even for convex domains in $\mathbf{R}^n$ Theorem 3 is an improvement of the usual version of Blaschke's result as it only requires that the principal curvatures are non-negative rather than positive. In the convex case this stronger version of the Blaschke theorem was first proven by Brooks and Strantzen [4, Thm 4.3.2 p53].

As a problem in Euclidean differential geometry it is as natural to roll a ball on the outside of a bounded non-convex domain as it is to roll it on the inside of a convex domain:

Theorem 4. *Let D be a bounded starlike open set in $\mathbf{R}^n$ with smooth boundary and let $M := \mathbf{R}^n \setminus D$ be the complement of D. Then M satisfies Blaschke's theorem* $\mathrm{Roll}(M) = \mathrm{Focal}(\partial D)$.

This is generalized in Theorem 3.1.

One use of the Blaschke theorem is to give lower bounds for the inradius of a domain $M \subset \mathbf{R}^n$. Under various assumptions on the topology of the domain M and assuming the principal curvatures of ∂M satisfy $|\lambda_i| \leq 1$

there are sharp lower bounds on the inradius of non-convex domains [20, 21, 22, 23, 24]. Several of these results have been extended to compact manifolds with boundary under curvatures bounds by Alexander and Bishop [1]. It is interesting that in most cases these lower bounds are less than $\mathrm{Focal}(\partial M)$ and thus do not follow from Blaschke's theorem or its generalizations.

As some readers may wish to avoid dealing with the formalism of Riemannian geometry in Section 5 we give more or less "tensor free" proofs of some of the results in the case M is a domain in Euclidean space. For other results and references relating to the rolling radius cf [2, 8, 11, 14, 19, 25, 26, 28] and especially [4] which has a wealth of information in the case of convex domains in $\mathbf{R}^n$.

2. The Main Lemma: Klingenberg Segments

In this section M will be a complete connected Riemannian manifold of dimension n with nonempty compact smooth boundary ∂M. In is not assumed the boundary is connected. Let ∇ be the Riemannian connection of M and R be the curvature tensor of M with the sign chosen so that $R(X,Y)Z = \nabla_X\nabla_Y Z - \nabla_Y\nabla_X Z - \nabla_{[X,Y]}Z$. For any vector X, the Ricci tensor Ric is defined by $\mathrm{Ric}(X,X) := \sum_{i=1}^{n-1}\langle R(X,e_i)e_i, X\rangle$ where $e_1,\dots,e_{n-1}$ is an orthonormal basis of $X^\perp$.

Let $\mathbf{n}$ be the inward pointing unit normal to ∂M, and let $I\!I$ be the second fundamental form of ∂M with respect to $\mathbf{n}$. That is for vector fields X, Y tangent to M, then $I\!I(X,Y) = \langle \nabla_X Y, \mathbf{n}\rangle$. The ***Weingarten map*** or ***shape operator*** of ∂M is the linear map on tangent spaces to ∂M given by $AX := -\nabla_X \mathbf{n}$. It is related to $I\!I$ by $\langle AX, Y\rangle := I\!I(X,Y)$. The mean curvature H of ∂M with respect to $\mathbf{n}$ at x is $H = (1/(n-1))\,\mathrm{trace}(A) = (1/(n-1))\sum_{i=1}^{n-1} I\!I(e_i,e_i)$ where $e_1,\dots,e_{n-1}$ is an orthogonal basis of $T(\partial M)_x$. The ***principal curvatures*** of ∂M with respect to $\mathbf{n}$ are the eigenvalues $\lambda_1,\dots,\lambda_{n-1}$ of the shape operator A.

For each $x \in \partial M$ set $\gamma_x(s) := \exp(s\mathbf{n}(x))$, the inward pointing unit speed geodesic normal to ∂M. Then the ***cut distance*** of ∂M along γ_x, denoted by $\mathrm{CutDist}_{\partial M}(x)$, is the supremum of the numbers s so that $\mathrm{dist}(\gamma_x(s), \partial M) = s$. The ***cut point*** of ∂M along γ_x is $\gamma_x(\mathrm{CutDist}_{\partial M}(x)\mathbf{n}(x))$. The ***cut locus*** of ∂M in M is the set $\mathrm{Cut}_{\partial M} := \{\gamma_x(\mathrm{CutDist}_{\partial M}(x)\mathbf{n}(x)) : x \in \partial M\}$ of all cut points of ∂M. If M is compact, then $\mathrm{CutDist}_{\partial M}(x)$ is bounded form above. However if M is not compact, then $\mathrm{CutDist}_{\partial M}(x) = \infty$ for at least one $x \in \partial M$. The ***rolling radius*** of M is $\mathrm{Roll}(M) := \min_{x\in\partial M}\mathrm{CutDist}_{\partial M}(x)$. There are natural cases, for example when M is the complement of an open bounded convex subset of $\mathbf{R}^n$, where $\mathrm{Roll}(M) = \infty$. The ***inradius*** of M is $\mathrm{InRad}(M) = \sup_{y\in M}\mathrm{dist}(y,\partial M)$. Clearly $\mathrm{Roll}(M) \leq \mathrm{InRad}(M)$.

The ***focal distance***, $\mathrm{focal}_{\partial M}(x)$, of ∂M along γ_x is supremum of the set of r so that the function defined on ∂M by $y \mapsto \mathrm{dist}(y, \gamma_x(r))$ has a local minimum at x. From the definitions $\mathrm{CutDist}_{\partial M}(x) \leq \mathrm{focal}_{\partial M}(x)$. The ***focal***

distance of ∂M is $\mathrm{Focal}(\partial M) := \min_{x\in\partial M}\mathrm{focal}_{\partial M}(x)$. There is a useful description of the focal distances in terms of Jacobi fields, which are in turn controlled by curvature. For $x \in \partial M$ a ∂M ***adapted Jacobi field*** V along γ_x is a vector field $s \mapsto V(s)$ that satisfies

$$V'' + R(V,\gamma_x')\gamma_x' = 0 \quad \text{and} \quad V(0) \in T(\partial M)_x, \quad V'(0) = -AV(0)$$

where $V' = \nabla V/ds$ and $V'' = \nabla^2 V/ds^2$. The point $\gamma_x(s_0)$ is a ***focal point*** of ∂M along γ_x iff there is an ∂M adapted Jacobi field V along γ_x so that $V(s_0) = 0$. If $\gamma_x(s_0)$ is the first focal point along γ_x then $\mathrm{focal}_{\partial M}(x) = s_0$. The first two parts of the following lemma are an easy variant on standard result characterizing the cut locus of a point in a Riemannian manifold cf [5, Lemma 5.2]. We give the short proof for completeness.

Proposition 2.1. *If $x \in \partial M$, then $\gamma_x(s_0)$ is the cut point of ∂M along γ_x if and only if one of the following holds for $s = s_0$ and neither holds for any smaller value of s:*

1. *$\gamma_x(s_0)$ is a focal point of ∂M along γ_x, or*
2. *There is anther geodesic segment $\alpha \neq \gamma_x$ form ∂M to $\gamma_x(s_0)$ so that α is perpendicular to ∂M and $\mathrm{Length}(\alpha) = s_0$.*

Moreover if $x \in \partial M$ and $z := \gamma_x(s_0)$ is the cut point of ∂M along γ_x and

3. *z is not a focal point of ∂M,*
4. *there are exactly two geodesic segments $\alpha_1 := \gamma_x$ and α_2, form z to ∂M with $\mathrm{Length}(\alpha_1) = \mathrm{Length}(\alpha_2) = \mathrm{dist}(z, \partial M)$,*

then near z the cut locus $\mathcal{C}_{\partial M}$ of ∂M is a smooth hypersurface and the tangent space $T(\mathcal{C}_{\partial M})_z$ to $\mathcal{C}_{\partial M}$ at z is the bisector of the two vectors $\alpha_2'(s_0)$ and $\alpha_2'(s_0)$.

Proof. Let $s_k > s_0$ be a sequence of real numbers with $s_k \searrow s_0$. Then as s_0 is the cut distance of ∂M along γ_x the distance of $\gamma_x(s_k)$ to ∂M is $< s_k$. Let $x_k \in \partial M$ be a point realizing the distance of $\gamma_x(s_k)$ to ∂M and let α_k be the minimizing geodesic form x_k to $\gamma_x(s_k)$. Then each $\alpha_k \neq \gamma_x$ and is orthogonal to ∂M. By passing to a subsequence it can be assumed the sequence $\{x_k\}$ converges, say $x_k \to x_\infty \in \partial M$. There are two cases, first $x_\infty = x$, in which case $\gamma_x(s_0)$ is a focal point of ∂M along γ_x. The second case is $x_\infty \neq x$. Then $\lim_{k\to\infty} \alpha_k =: \alpha$ is a geodesic segment form ∂M to $\gamma_x(s_0)$ of length s_0 and perpendicular to ∂M.

To prove parts 3 and 4 imply the last part of the proposition set $x_1 = x$, let $x_2 \in \partial M$ be the initial point of the geodesic α_2 and let $U_i \subset \partial M$ be a very small open neighborhood of x_i in ∂M. Let $\rho_i \colon M \to [0,\infty)$ be $\rho_i(y) := \mathrm{dist}(y, U_i)$. As ∂M has no focal points along the geodesic segment α_i the function ρ_i is smooth in a neighborhood of z. Let $\overline{B}(z,r)$ be the closed geodesic ball of radius r centered at z. Then $\overline{B}(z,s_0)\cap\partial M = \{x_1, x_2\}$. Therefore there is an r just little larger than s_0 so that $\overline{B}(z,r)\cap\partial M \subset U_1\cup U_2$. If $\delta := r - s_0$ and $p \in \overline{B}(z,\delta)$, then the point $q \in \partial M$ closest to p is in $U_1 \cup U_2$, for

$$\mathrm{dist}(q,p) \le \mathrm{dist}(x_1,p) \le \mathrm{dist}(x_1,z) + \mathrm{dist}(z,p) \le s_0 + \delta \le r.$$

So that $q \in \overline{B}(x, s_0) \cap \partial M \subset U_1 \cup U_2$. By possibly making δ smaller it can also be assumed $\overline{B}(z, \delta)$ contains no focal points of ∂M, so that for each $p \in \overline{B}(z, \delta)$ there is a unique point in each of U_1 and U_2 closest to p.

We now claim $\mathcal{C}_{\partial M} \cap B_\delta(z, \delta) = \{p : \rho_1(p) = \rho_2(p)\}$. To see this note no point of $\overline{B}(z, \delta)$ is a focal point of ∂M. Thus by the first part of the proposition the only way a point $p \in \overline{B}(z, \delta)$ can be in the cut locus is if there are two or more minimizing geodesic geodesic segments from p to ∂M. By the choice of δ this means that there are exactly two geodesic segments form p to ∂M, one with an endpoint in U_1 and one with an end point in U_2 and of the same length. But this set of points is $\{p : \rho_1(p) = \rho_2(p)\}$.

The function $f := \rho_2 - \rho_1$ is smooth in $\overline{B}(z, \delta)$. The gradient of ρ_i at $z = \alpha_i(s_0)$ is $\nabla \rho_i(z) = \alpha_i'(s_0)$. Thus $\nabla f(z) = \nabla \rho_2(z) - \nabla \rho_1(z) = \alpha_2'(s_0) - \alpha_1'(s_0)$. If this were zero, then $\alpha_2'(s_0) = \alpha_1'(s_0)$, which would imply that $\alpha_1 = \alpha_2$ which is not the case. Thus $\nabla f(z) \neq 0$. Therefore by the implicit function theorem the set $\{p : \rho_1(p) = \rho_2(p)\}$ is a smooth hypersurface of M near z with normal vector $\nabla f(z) = \alpha_2'(s_0) - \alpha_1'(s_0)$. This implies $T(\mathcal{C}_{\partial M})_z$ is the hyperplane of $T(M)_z$ bisecting $\alpha_1'(s_0)$ and $\alpha_2'(s_0)$. □

Proposition 2.2 (Main Lemma)**.** *Let M be a complete Riemannian manifold with non-empty compact boundary ∂M. Assume $R_0 := \mathrm{Roll}(M) < \mathrm{Focal}(\partial M)$. Then there is a geodesic segment $\gamma \colon [0, 2R_0] \to M$ so that*

1. *γ is orthogonal to ∂M at both its end points $\gamma(0)$ and $\gamma(2R_0)$.*
2. *The midpoint $z = \gamma(R_0)$ of γ is in the cut locus $\mathcal{C}_{\partial M}$ and near z $\mathcal{C}_{\partial M}$ is a smooth hypersurface orthogonal to γ at z.*
3. *There is a neighborhood $\mathcal{N}$ of γ in the C^0 topology so that any $c \in \mathcal{N}$ with both endpoints on ∂M satisfies $\mathrm{Length}(c) \geq \mathrm{Length}(\gamma) = 2R_0$. (In particular the second variation of arclength of any smooth variation of γ with endpoints on ∂M is non-negative and ∂M has no focal points along γ.)*

*When $\mathrm{Roll}(M) < \mathrm{Focal}(\partial M)$ any segment $\gamma \colon [0, 2R_0] \to M$ satisfying these three conditions will be called a **Klingenberg segment**.*

Remark 2.3. These lemmas and their proofs are motivated by results of Klingenberg [17, 18] about the cut and conjugate loci of points in Riemannian manifolds. Klingenberg showed if the injectivity radius, $\mathrm{Inject}(M)$, of a compact Riemannian manifold M is less than its conjugate distance, $\mathrm{Conj}(M)$, then $\mathrm{Inject}(M)$ is one half the length of the shortest closed geodesic in M. The proposition above shows if $\mathrm{Roll}(M) < \mathrm{Focal}(\partial M)$ then $\mathrm{Roll}(M)$ is half the length of a Klingenberg segment. Thus when dealing with manifolds with boundary $\mathrm{Roll}(M)$ and $\mathrm{Focal}(\partial M)$ are analogous to $\mathrm{Inject}(M)$ and $\mathrm{Conj}(M)$ for manifolds without boundary. For a farther similarity between $\mathrm{Roll}(M)$ and $\mathrm{Inject}(M)$ note the map $(x, t) \mapsto \exp(t\mathbf{n}(x))$ is a diffeomorphism from $\partial M \times [0, R_0)$ to a tubular neighborhood of ∂M in ∂M when $R_0 = \mathrm{Roll}(M)$ but this is not true for any $R_0 > \mathrm{Roll}(M)$. Likewise

$\mathrm{Inject}(M)$ is the largest R_1 so for all $x \in M$ and the map $\exp_x\colon T(M)_x \to M$ is a diffeomorphism on the open ball of radius R_1 in $T(M)_x$.

Proof. Let $z \in \mathcal{C}_{\partial M}$ be a point of $\mathcal{C}_{\partial M}$ with $\mathrm{dist}(z, \partial M) = R_0$ and let $\alpha_1\colon [0, R_0] \to M$ be a geodesic segment from ∂M to z so that $z = \alpha_1(R_0)$ is the cut point of ∂M along γ_1. As $\mathrm{Roll}(M) < \mathrm{Focal}(\partial M)$ the point z is not a focal point of ∂M. Therefore by the last proposition there is at least one other geodesic segment $\alpha_2\colon [0, R_0] \to M$ from ∂M to z. As both α_1 and α_2 minimize the distance to ∂M they are perpendicular to ∂M. We now use the notation of the proof of the last proposition. Let $x_i \in \partial M$ be the initial point of α_i and $U_i \subset \partial M$ a very small open neighborhood of x_i in ∂M. Again let $\rho_i(p) = \mathrm{dist}(p, U_i)$. As in the last proof both ρ_1 and ρ_2 are smooth functions in a neighborhood of z, the function $f := \rho_2 - \rho_1$ has a non-zero gradient at z so that $N := \{p : \rho_1(p) = \rho_2(p)\}$ is a smooth hypersurface near z whose tangent space $T(N)_z$ is the bisector between of the vectors $\nabla\rho_1(z) = \alpha_1'(R_0)$ and $\nabla\rho_2(z) = \alpha_2'(R_0)$.

If $\nabla\rho_2(z) = \alpha_2'(R_0) \neq -\alpha_1'(R_0) = -\nabla\rho_1(z)$ then there is curve β in N with $\beta(0) = z$ and with $d\rho_1(c'(0)) = \langle c'(0), \alpha_1'(R_0)\rangle > 0$ and $d\rho_2(c'(0)) = \langle c'(0), \alpha_2'(R_0)\rangle > 0$. This implies near $t = 0$ both $\rho_1(\beta(t))$ and $\rho_2(\beta(t))$ are decreasing functions of t. Thus for small $t > 0$ we have $\rho_1(\beta(t)) = \rho_2(\beta(t)) < R_0$. Fix such a t and let $R_1 := \rho_1(\beta(t)) = \rho_2(\beta(t)) < R_0$. Then there are geodesic segments σ_i of length R_1 and orthogonal to ∂M at a point of U_i. By the last lemma this implies that the cut distance of ∂M along σ_i is $\leq R_1 < R_0$. But this contradicts that $R_0 = \mathrm{Roll}(M)$ is the smallest cut distance of ∂M. Therefore the assumption $\alpha_2'(R_0) \neq -\alpha_1'(R_0)$ leads to a contradiction and so $\alpha_2'(R_0) = -\alpha_1'(R_0)$. Define $\gamma\colon [0, 2R_0] \to M$ by $\gamma(t) := \alpha_1(t)$ for $0 \leq t \leq R_0$ and $\gamma(t) := \alpha_2(2R_0 - t)$ for $R_0 \leq t \leq 2R_0$. Then γ pieces together to be a smooth geodesic with midpoint $z = \gamma(R_0)$ and perpendicular to ∂M at both end points.

If $\alpha \neq \alpha_1$ is any geodesic segment from ∂M to z of length R_0 then the argument just given yields $\alpha'(R_0) = -\alpha_1'(R_0)$ which implies $\alpha = \alpha_2$. Thus there are only two minimizing geodesics form z to ∂M and z is not a focal point of ∂M so by the last proposition the cut locus $\mathcal{C}_{\partial M}$ is a smooth hypersurface near z whose tangent space is the bisector between $\alpha_1'(R_0)$ and $\alpha_2'(R_0) = -\alpha_1'(R_0)$, which is equivalent to being perpendicular to γ. This completes the proof of the first part of the proposition.

To prove the last part, as $\gamma \perp \mathcal{C}_{\partial M}$ is at z there is a neighborhood $\mathcal{N}$ of γ in the C^0 topology so that any $c \in \mathcal{N}$ intersects $\mathcal{C}_{\partial M}$ in a point near z. Let $c \in \mathcal{N}$ have both endpoints on ∂M. Then c intersects $\mathcal{C}_{\partial M}$ and as $\mathrm{dist}(\mathcal{C}_{\partial M}, \partial M) = R_0$ clearly $\mathrm{Length}(c) \geq 2R_0 = \mathrm{Length}(\gamma)$. This completes the proof. □

3. Complements of Bounded Domains

One consequence of the Main Lemma 2.2 is that if M is a complete Riemannian manifold with non-empty compact boundary with the property that no geodesic segment of M is perpendicular to ∂M at both of its endpoints, then $\operatorname{Roll}(M) = \operatorname{Focal}(\partial M)$. Here we give examples where this condition holds. Let N be a complete simply connected Riemannian manifold with no boundary and all sectional curvatures ≤ 0. Then by the Cartan-Hadamard theorem [15, §2.1 p187] N is diffeomorphic to $\mathbf{R}^n$ and through any two points of N there is exactly one geodesic. An open subset D of N is ***starlike*** iff there a point $x_0 \in U$ so that every point of D can be connected to N by a geodesic segment staying inside of U.

Theorem 3.1. *Let N be a complete simply connected Riemannian manifold of non-positive sectional curvature and empty boundary. Let $D \subset N$ be a bounded open subset of N with smooth connected boundary so that either*

1. *D is starlike with respect to some point, or*
2. *$\dim N = 2$ and the geodesic curvature of ∂D satisfies $\displaystyle\int_{\partial D} |\kappa|\, ds < 4\pi$.*

Then the manifold $M := N \setminus D$ satisfies Blaschke's theorem $\operatorname{Roll}(M) = \operatorname{Focal}(\partial M)$.

Proof. We first assume D is starlike with respect to a point x_0. If $M \setminus D$ does not satisfy $\operatorname{Roll}(M) = \operatorname{Focal}(\partial M)$, then $R_0 := \operatorname{Roll}(M) < \operatorname{Focal}(\partial M)$ and by Proposition 2.2 there is a Klingenberg segment $\gamma\colon [0, 2R_0] \to M$ orthogonal to $\partial M = \partial D$ at its endpoints $x_1 := \gamma(0)$ and $x_2 := \gamma(2R_0)$. For $i = 1, 2$ let γ_i be the geodesic segment of N form x_0 to x_i. As D is starlike with respect to x_0 all the points of γ_i except x_i are in D. As γ is perpendicular to ∂D at x_i this implies the angle between γ and γ_i satisfies $\sphericalangle(\gamma_i, \gamma) \geq \pi/2$. Because the sectional curvatures of N are ≤ 0 one of the standard triangle comparison theorems (see [15, Thm 4.1 p197]) there is a triangle T in the Euclidean plane $\mathbf{R}^2$ with sides $\overline{\gamma}$, $\overline{\gamma}_1$ $\overline{\gamma}_2$ so that $\sphericalangle(\overline{\gamma}, \overline{\gamma}_i) \geq \sphericalangle(\gamma, \gamma_i) \geq \pi/2$. This implies the sum of the angles of T is greater than π which is impossible for a triangle in $\mathbf{R}^2$. This contradiction implies $\operatorname{Roll}(M) = \operatorname{Focal}(M)$ must hold.

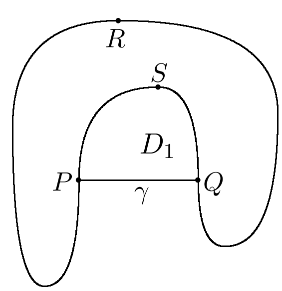

FIGURE 1.

Now assume $\dim N = 2$ and that N is complete, simply connected, and has Gaussian curvature $K \leq 0$. Assume D is a bounded domain in N with smooth boundary and so that the geodesic curvature of the boundary satisfies $\int_{\partial D}|\kappa|\,ds < 4\pi$. Let $M = N\setminus D$. Again if $R_0 := \mathrm{Roll}(M) < \mathrm{Focal}(\partial M)$ by Proposition 2.2 there is a Klingenberg segment $\gamma\colon [0,2R_0]\to M$ that is orthogonal to ∂M at both of its endpoints. Let $P := \gamma(0)$ and $Q := \gamma(2R_0)$. These points divide the curve $\partial M = \partial D$ into two arcs $\widehat{PRQ}$ and $\widehat{PSQ}$ (cf. Figure 1). Then $\gamma\cup\widehat{PSQ}$ is a simple closed curve and N is diffeomorphic to the plane $\mathbf{R}^2$ so by the Jordan curve theorem $N\setminus(\gamma\cup\widehat{PSQ})$ has a bounded component D_1. Likewise let D_2 be the bounded component of $N\setminus(\gamma\cup\widehat{PRQ})$. Let α_1 and α_2 be the two interior angles of D_1 and β_1 and β_2 the two interior angles of D_2. Then as γ is perpendicular to ∂D all of exactly two of the four angles $\alpha_1,\alpha_2,\beta_1,\beta_2$ are equal to $\pi/2$ and the other two are equal to $3\pi/2$. We now claim that α_1 and α_2 are either both $\pi/2$ or $3\pi/2$. For if $\alpha_1 = 3\pi/2$ and $\alpha_2 = \pi/2$ and the angle α_1 is at the point P, then the part of the arc $\widehat{PRQ}$ near P is inside of D_1, but the part of $\widehat{PRQ}$ near Q is outside of D_1. This would imply that $\widehat{PRQ}$ intersects $\partial D_1 = \gamma\cup\widehat{PSQ}$ at a point other than one of its endpoints which is impossible.

Choose the notation so that D_1 has both its interior angles equal to $\pi/2$ and D_2 has both its interior angles equal to $3\pi/2$. If U is a bounded simply connected domain in with piecewise smooth boundary, interior angles θ_i and boundary geodesic curvature κ then the Gauss-Bonnet theorem [6, p125] gives

$$\int_{\partial U}\kappa\,ds = 2\pi + \sum_i(\theta_i-\pi) - \int_D K\,dA \geq 2\pi + \sum_i(\theta_i-\pi)$$

as $K\leq 0$ on N. As both interior angles of D_1 are $\pi/2$ and the interior angles of D_2 are $3\pi/2$

$$\int_{\partial D_1}|\kappa|\,ds = \int_{\widehat{PSQ}}|\kappa|\,ds \geq \pi, \qquad \int_{\partial D_2}|\kappa|\,ds = \int_{\widehat{PRQ}}|\kappa|\,ds \geq 3\pi,$$

so that $\int_{\partial D}|\kappa|\,ds = \int_{\widehat{PSQ}}|\kappa|\,ds + \int_{\widehat{PRQ}}|\kappa|\,ds \geq 4\pi$, a contradiction. This completes the proof. □

4. Curvature and Blaschke's Theorem

Define $\mathrm{c}_{K_0}(s)$ and $\mathrm{s}_{K_0}(s)$ by the initial value problems

$$\begin{aligned}
\mathrm{c}''_{K_0}(s) + K_0\mathrm{c}_{K_0}(s) &= 0, &\quad \mathrm{c}_{K_0}(0) = 1, &\quad \mathrm{c}'_{K_0}(0) = 0,\\
\mathrm{s}''_{K_0}(s) + K_0\mathrm{s}_{K_0}(s) &= 0, &\quad \mathrm{s}_{K_0}(0) = 0, &\quad \mathrm{s}'_{K_0}(0) = 1.
\end{aligned}$$

Explicitly:

$$\mathrm{c}_{K_0}(s) = \begin{cases} \cos\left(\dfrac{s}{\sqrt{K_0}}\right), & K_0 > 0\\ 1, & K_0 = 0\\ \cosh\left(\dfrac{s}{\sqrt{-K_0}}\right), & K_0 < 0\end{cases}$$

$$\mathrm{s}_{K_0}(s) = \begin{cases} \sqrt{-K_0}\,\sin\left(\dfrac{s}{\sqrt{K_0}}\right), & K_0 > 0 \\ s, & K_0 = 0 \\ \sqrt{-K_0}\,\sinh\left(\dfrac{s}{\sqrt{-K_0}}\right), & K_0 < 0. \end{cases}$$

Definition 4.1. Let M be a Riemannian manifold with boundary. Then the ***normal width*** of M is $\geq W_0$ iff every geodesic $\gamma_x(s)$ with $x \in \partial M$ and $\gamma_x'(0) = \mathbf{n}(x)$ meets ∂M for the first time at some point $\gamma_x(s_0)$ with $s_0 \geq W_0$. The maximum such W_0 is the normal width of M and will be denoted by $\mathrm{NorWid}(M)$. □

Remark 4.2. If M is a convex domain in $\mathbf{R}^n$ then $\mathrm{NorWid}(M)$ is the width in the usual sense (that is the smallest distance W_0 so that M in the region between two parallel hyperplanes at a distance D_0 form each other). However if $M \subset \mathbf{R}^n$ is not convex then generally $\mathrm{NorWid}(M)$ will be less than the width. The quantity $\mathrm{NorWid}(M)$ is interesting in extending Blaschke's theorem to positively curved manifolds. Theorem 1 of the introduction implies Blaschke's theorem holds when M is a domain in a hemisphere of the unit sphere provided ∂M has non-negative mean curvature. We will show that for domains on the sphere with $\mathrm{NorWid}(M) \geq \pi$ that $\mathrm{Roll}(M) = \mathrm{Focal}(\partial M)$ without any assumption on the boundary curvature.

Theorem 4.3. *Let M be a complete connected n dimensional Riemannian manifold with non-empty compact boundary ∂M. Assume the Ricci tensor of M and the mean curvature H of ∂M with respect to the inward normal satisfy*

$$\mathrm{Ric}_M \geq (n-1)K_0, \quad H \geq H_0,$$

and that at least one of the following holds

1. $K_0 \leq 0$ *and* $H_0 \geq \sqrt{-K_0}$, *or*
2. $K_0 > 0$ *and* $\mathrm{NorWid}(M) \geq \dfrac{\pi}{\sqrt{K_0}}$ *(with no condition on the mean curvature).*

Then $\mathrm{Roll}(M) < \mathrm{Focal}(\partial M)$ *if and only if* $K_0 = H_0 = 0$ *and M is either a Riemannian cylinder or a generalized Möbius band.*

Theorem 4.4. *Let M be a compact n dimensional Riemannian manifold with non-empty smooth boundary ∂M. Assume the sectional curvature of M and the principal curvatures $\lambda_1, \dots, \lambda_{n-1}$ of ∂M with respect to the inward normal satisfy:*

$$K_M \geq K_0, \quad \text{at any point of } \partial M \text{ at least } n/2 \text{ of the } \lambda_i \text{ satisfy } \lambda_i \geq H_0.$$

Also assume:

1. *If* $K_0 = 0$, *then* $H_0 > 0$.
2. *If* $K_0 < 0$, *then* $H_0 \geq \sqrt{-K_0}$.

3. *If* $K_0 > 0$, *then* $\operatorname{NorWid}(M) \ge \dfrac{\pi}{\sqrt{K_0}}$ *(with no restrictions on the principal curvatures).*

Then $\operatorname{Roll}(M) = \operatorname{Focal}(\partial M)$.

Proof of Theorem 4.3. We first show if M is non-compact $\operatorname{Roll}(M) = \operatorname{Focal}(\partial M) = \infty$. If $K_0 > 0$ then M is compact because the first focal point of ∂M along an inward pointing normal geodesic γ at $x \in \partial M$ is at a distance less than the first conjugate distance of x along $\gamma \le \rho_0$ and the first conjugate distance is $\le \pi/\sqrt{K_0}$ (cf [5, Thm 1.26 p27]). Thus every point of M is at a distance $\le \pi/\sqrt{K_0}$ form ∂M. As ∂M is compact this implies M is compact. If $K_0 \le 0$ and M is non-compact, then the warped product splitting theorems of Kasue [16, Thm C] or Croke-Kleiner [7, Thm 3] apply and M is isometric to either a product (if $K_0 = 0$) or a warped product (if $K_0 < 0$) of ∂M and a ray $[0,\infty)$. In either case $\operatorname{Roll}(M) = \operatorname{Focal}(\partial M) = \infty$ as claimed.

Therefore assume M is compact. First consider the case $K_0 > 0$ and $\operatorname{NorWid}(M) \ge \pi/\sqrt{K_0}$. If $\gamma_x(s) = \exp_x(s\mathbf{n}(x))$ is a geodesic normal to ∂M and l is so that $\gamma_x(l) \in \partial M$ then $l \ge \pi/\sqrt{K_0}$. As the first focal point of ∂M along γ_x occurs before the first conjugate point of x along γ_x as above this means the first focal point of ∂M along γ_x occurs at a point $\gamma(a)$ with $a < l$. Therefore by part 3 of Proposition 2.2 γ_x is not a Klingenberg segment. But if M has no Klingenberg segments then Proposition 2.2 implies $\operatorname{Roll}(M) = \operatorname{Focal}(\partial M)$.

This leaves the cases where $K_0 \le 0$. We first prove the result under the more restrictive conditions

$$K_0 = 0,\ H_0 > 0, \quad \text{or} \quad K_0 < 0,\ H_0 \ge \sqrt{-K_0}. \tag{4.1}$$

(Strictly speaking the proof of this theorem only requires us to consider the case of $H_0 \le \sqrt{-K_0}$, however the calculations in the other case are needed in the proof of Theorem 4.4.) If $R_0 := \operatorname{Roll}(M) < \operatorname{Focal}(\partial M)$ by Proposition 2.2 there is Klingenberg segment $\gamma\colon [0, 2R_0] \to M$. To simplify notation set $l := 2R_0$. By a ***smooth variation*** of γ we mean a smooth map $\alpha\colon [0,l] \times (-\varepsilon,\varepsilon) \to M$ so that $\alpha(s,0) = \gamma(s)$ and for all $t \in (-\varepsilon,\varepsilon)$ $\alpha(0,t) \in \partial M$, and $\alpha(l,t) \in \partial M$. Under the assumptions (4.1) we show there is a smooth variation α of γ so that $\operatorname{Length}(s \mapsto \alpha(s,t)) < l$ for small $t \ne 0$. This is based on the second variation formula for arclength. Given a smooth variation α of γ, let α_t be the curve $s \mapsto \alpha(s,t)$ and let $L(t) := \operatorname{Length}(\alpha_t)$. Let $V(s) := \partial\alpha/\partial t(s,0)$ be the variation vector field of α and let $V' := \nabla V/\partial s$, $V'' := \nabla^2 V/\partial s^2$ be the covariant derivatives of V along γ. Then the second variation formula [9, p208] is

$$L''(0) = \int_0^l \left(\langle V', V'\rangle - \langle R(V,\gamma')\gamma', V\rangle\right) ds - I\!I(V(0),V(0)) - I\!I(V(l),V(l)).$$

Let $e_i(s)$ $(1 \le i \le n-1)$ be parallel vector fields along γ so that for each s the vectors $e_1(s), \dots, e_{n-1}(s)$, $\gamma'(s)$ are an orthonormal basis of $T(M)_{\gamma(s)}$. Let f be a smooth function on $[0,l]$ and set $V_i(s) = f(s)e_i(s)$. Then $V' =$

$f'e_i$. Let α_i be smooth variations of γ so that $\nabla\alpha_i/\partial t(s,0) = V_i(s)$ and set $L_i(t) = \operatorname{Length}((\alpha_i)_t)$. Averaging the second variations of the α_i's and using the bounds on the Ricci and mean curvature along with integration by parts:

$$\begin{aligned}\frac{1}{n-1}\sum_{i=1}^{n-1} L_i''(0) &= \int_0^l \left((f')^2 - \frac{1}{n-1}\operatorname{Ric}(\gamma',\gamma')\right) ds \\ &\qquad - f(0)^2 H(\gamma(0)) - f(l)^2 H(\gamma(l)) \\ &\le \int_0^l ((f')^2 - K_0 f^2)\, ds - H_0(f(0)^2 + f(l)^2) \\ &= -f(0)(f'(0) + H_0 f(0)) + f(l)(f'(l) - H_0 f(l)) \\ &\qquad - \int_0^l f(f'' + K_0 f)\, ds\end{aligned}$$

Let $f(s) = \mathrm{c}_{K_0}(s-l/2)$. Using $\mathrm{c}_{K_0}' = K_0\mathrm{s}_{K_0}$, $\mathrm{c}_{K_0}(-s) = \mathrm{c}_{K_0}(s)$, $\mathrm{s}_{K_0}(-s) = -\mathrm{s}_{K_0}(s)$, and $f'' + K_0 f = 0$, for this choice of f the last inequality becomes

$$\frac{1}{n-1}\sum_{1}^{n-1} L_i''(0) \le -2\mathrm{c}_{K_0}(l/2)(H_0\mathrm{c}_{K_0}(l/2) - K_0\mathrm{s}_{K_0}(l/2))$$

We now claim under the assumptions of (4.1) this is negative. In the, $K_0 = 0$, $H_0 > 0$, we have $\mathrm{c}_{K_0} \equiv 1$ so the inequality becomes $\frac{1}{n-1}\sum_{i=1}^{n-1} L_i''(0) \le -2H_0 < 0$. In the case $K_0 < 0$ and $H_0 \ge \sqrt{-K_0}$ the inequality reduces to

$$\begin{aligned}\frac{1}{n-1}\sum_{i=1}^{n-1} L_i''(0) \le -2\mathrm{c}_{K_0}(l/2)\Big(&H_0\cosh(\sqrt{-K_0}\,(l/2)) \\ &-\sqrt{-K_0}\sinh(\sqrt{-K_0}\,(l/2))\Big) < 0\end{aligned}$$

as $\cosh(t) > \sinh(t) > 0$ for all $t > 0$. Thus $\frac{1}{n-1}\sum_{i=1}^{n-1} L_i''(0) < 0$ in this case also. But if the average of the variations $L_i''(0)$ is negative, then for at least one i the second variation of α_i is negative. Thus $L_i(t) < L(0) = \operatorname{Length}(\gamma) = l$ for small $t \neq 0$ which contradicts part 3 of Proposition 2.2. Therefore $\operatorname{Roll}(M) = \operatorname{Focal}(\partial M)$ under the assumptions of (4.1).

In general case of $K_0 = 0$ and $H_0 \ge 0$ assume γ is a Klingenberg segment as above and that U_1 is a small neighborhood of $\gamma(0)$ in ∂M and U_2 is a small neighborhood of $\gamma(l)$ in ∂M. Let $\rho_i \colon M \to [0,\infty)$ be $\rho_i(x) := \operatorname{dist}(x, U_i)$. Then Proposition 2.2 implies that in a small tubular neighborhood $\tau_\delta(\gamma)$ of γ that $\rho_1 + \rho_2 \ge l = \operatorname{Length}(\gamma)$. Therefore we are in the case of a "local" version of the warped product splitting theorems [16, 7] and the same arguments apply. By basic comparison results [16, Lemma 1.1] both ρ_1 and ρ_2 are sub-harmonic in a neighborhood of γ. Thus $\rho_1 + \rho_2$ is also sub-harmonic in a neighborhood of γ and for any point $\gamma(t)$ of γ there holds $\rho_1(\gamma(t)) + \rho_2(\gamma_2(t)) = t + (l-t) = l$. Then $\rho_1 + \rho_2$ has a local minimal at $\gamma(t)$ and so by the maximal principal [13, Thm 3.5] $\rho_1 + \rho_2 \equiv l$ is constant in a neighborhood of γ. Therefore $\rho_1 \equiv l$ on U_2. By the argument in the proof of Theorem 1 of [7] this implies that a tubular neighborhood of γ splits isometrically as product $U_1 \times [0, l]$. An easy continuation argument now shows that near every point $x \in \partial M$ that in a neighborhood of $\gamma_x(t) := \exp(t\mathbf{n}(x))$ M splits locally as a metric product $U \times [0, l]$ for some neighborhood of x in ∂M and the cut locus $\mathcal{C}_{\partial M}$ is the set $\{\exp(R_0\mathbf{n}(x)) : x \in \partial M\}$ of midpoints of the segments normal to M. Moreover the map $x \mapsto \exp(R_0\mathbf{n}(x))$ from ∂M to $\mathcal{C}_{\partial M}$ is a two to one local isometry. The cut locus $\mathcal{C}_{\partial M}$ is a deformation retract of M (the homotopy is accomplished by moving in along the normal to ∂M) and M is connected therefore $\mathcal{C}_{\partial M}$ is connected. As $x \mapsto \exp(R_0\mathbf{n}(x))$ is two to one and $\mathcal{C}_{\partial M}$ is connected the boundary consists either of two connected components isometric to $\mathcal{C}_{\partial M}$ or ∂M is a double cover of $\mathcal{C}_{\partial M}$. As M is locally a product of $\mathcal{C}_{\partial M}$ and an interval $[0, l] = [0, 2R_0]$ it follows if ∂M has two components then M is isometric to the product $\partial M \times [0, l]$. If ∂M is connected then define a map $\theta \colon \partial M \to \partial M$ by $\theta(x) := \exp(2R_0\mathbf{n}(x)) \in \partial M$. Because M is locally a product of ∂M and $[0, 2R_0]$ this map is a fixed point free isometry of ∂M. Let $\sim$ be the equivalence relation on $\partial M \times [0, R_0]$ that identifies (x, R_0) with $(\theta(x), R_0)$ so that $\partial M \times [0, R_0]/\sim$ is a generalized Möbius band. The map $\Theta(x, t) := \exp(t\mathbf{n}(x))$ form $\partial M \times [0, R_0]/\sim$ to M is then an isometry so M is a generalized Möbius band. $\square$

Proof of Theorem 4.4. First consider the case of $K_0 > 0$ and $\operatorname{NorWid}(M) \geq \pi/\sqrt{K_0}$. As in the beginning of the last proof, this implies any point of a geodesic has a conjugate point at distance $\leq \pi/\sqrt{K_0}$ (cf [5, Thm 1.26 p27]). Then the argument above shows M has no Klingenberg segments and thus $\operatorname{Roll}(M) = \operatorname{Focal}(\partial M)$.

When $K_0 \leq 0$ assume, toward a contradiction, $R_0 := \operatorname{Roll}(M) < \operatorname{Focal}(\partial M)$. By Proposition 2.2 there is a Klingenberg segment $\gamma \colon [0, 2R_0] \to M$. Set $l = 2R_0$, $x_1 := \gamma(0)$, and $x_2 := \gamma(l)$. By hypothesis there is a linear subspace $W_i \subset T(\partial M)_{x_i}$ with $\dim W_i \geq (\dim M)/2$ and $I\!I(X, X) \geq H_0\langle X, X\rangle$ for all $X \in W_i$. If W_1 is parallel translated along γ to $\gamma(l) = x_2$ by elementary linear algebra this translated subspace has a nonzero vector in common with W_2 (as $\dim W_0 + \dim W_1 \geq 2(n/2) > (n-1) = \dim(\partial M)$). Thus there is a parallel unit vector field $e(t)$ along $\gamma(t)$ so that $I\!I(e(0), e(0)) \geq H_0$ and $I\!I(e(l), e(l)) \geq H_0$. As in the proof of Theorem 4.3 let $f(s) = \mathrm{c}_{K_0}(s - l/2)$ and $V(t) = f(t)e(t)$. Using this V in the second variation formula and doing a calculation like the one in the proof of Theorem 4.3 leads to

$$L''(0) \leq -2\mathrm{c}_{K_0}(l/2)(K_0\mathrm{s}_{K_0}(l/2) + H_0\mathrm{c}_{K_0}(l/2)).$$

As in the proof of Theorem 4.3 this is negative which contradicts part 3 of Proposition 2.2. $\square$

5. Euclidean Proofs

The two basic Propositions 2.1 and 2.2 hold in the Euclidean with exactly the same proofs (just replace geodesics with lines and ignore the parts the first three paragraphs of Section 2 that deal with the Riemannian set up). The results of Section 3 simplify in the Euclidean case: It is more or less obvious that if $D \subset \mathbf{R}^n$ is a starlike bounded open set with smooth boundary then $M := \mathbf{R}^n \setminus D$ has no segments γ perpendicular to ∂M at both end points. Thus M has no Klingenberg segments and Proposition 2.2 implies $\mathrm{Roll}(M) = \mathrm{Focal}(\partial M)$.

Let $M \subset \mathbf{R}^n$ be a domain with ∂M smooth and connected. If $R_0 := \mathrm{Roll}(M) < \mathrm{Focal}(\partial M)$ then Proposition 2.2 yields a Klingenberg segment $\gamma\colon [0, 2R_0] \to M$. By a rotation and translation we assume γ is the segment between the points $\gamma(0) = (0,0) \in \mathbf{R}^{n-1} \times \mathbf{R}$ and $\gamma(2R_0) = (0, 2R_0) \in \mathbf{R}^{n-1} \times \mathbf{R}$. Let U_1 be a small open piece of ∂M near $(0,0)$ and U_2 a small open piece of ∂M near $(0, 2R_0)$. Let V be a small open neighborhood of the origin in $\mathbf{R}^{n-1}$ then both U_1 and U_2 can be expressed as the graphs of functions $f_i\colon V \to \mathbf{R}$. That is (after making U_1, U_2 and V smaller) we have $U_i = \{(x, f_i(x)) : x \in V\}$. As the segment γ is a part of the x_n-axis and γ is perpendicular ∂M for $i = 1, 2$ and $\alpha = 1, \dots, n-1$ we have $\partial f_i/\partial x_\alpha(0) = 0$. If $D^2 f_i(\cdot,\cdot)$ is the Hessian of f_i (viewed as a symmetric bilinear form whose matrix in the standard orthonormal basis $e_1, \dots, e_{n-1}$ of $\mathbf{R}^{n-1}$ is $[\partial^2 f_i/\partial x_\alpha \partial x_\beta]$), the second fundamental form $I\!I_i(\cdot,\cdot)$ of ∂M at the point $(0, f_i(0))$ is $I\!I_i(\cdot,\cdot) = (-1)^{i+1} D^2 f(\cdot,\cdot)$ (the factor of -1 for $I\!I_2$ comes from the fact the inward normal to ∂M at $(0, f_2(0))$ points downward). So the Taylor's expansion of f_i is

$$f_i(x) = f_i(x_1, \dots, x_{n-1}) = f_i(0) + \frac{(-1)^{i+1}}{2} \sum_{\alpha,\beta=1}^{n-1} I\!I_i(e_\alpha, e_\beta) x_\alpha x_\beta + O(\|x\|^3)$$

For $v \in \mathbf{R}^{n-1}$ let $c_{v,s}\colon [0, 2R_0] \to M$ be the line segment

$$c_{v,s}(t) := \left(sv, \frac{(2R_0 - t) f_1(sv) + t f_2(sv)}{2R_0} \right).$$

The length $L(c_{v,s})$ of the curve is the distance between its endpoints which is $f_2(sv) - f_1(sv)$. Using the Taylor expansion for f_i

$$(5.1) \qquad \frac{d}{ds} L(c_{v,s}) \Big|_{s=0} = 0, \qquad \frac{d^2}{ds^2} L(c_{v,s}) \Big|_{s=0} = -(I\!I_1(v,v) + I\!I_2(v,v)).$$

The principal curvatures of ∂M with respect to the inward normal $\mathbf{n}$ are the eigenvalues of the second fundamental form $I\!I$ of ∂M (with respect to the induced metric on ∂M). If at each point x of ∂M at least $n/2$ of the principal curvatures of ∂M are positive, then at each point x of ∂M, there is a subspace V of the tangent space of ∂M to ∂M at x with dimension $\geq n/2$ so that $I\!I$ is positive definite on V (take V to be the span of the eigenvectors of $I\!I$ corresponding to the positive principal curvatures). Let $V_i \subseteq \mathbf{R}^{n-1}$ be a subspace of $\mathbf{R}^{n-1}$ of dimension $\geq n/2$ with $I\!I_i$ positive definite on V_i. Then $\dim V_1 + \dim V_2 > n-1$ so there is a vector $0 \neq v \in V_1 \cap V_2$. Using this vector in (5.1) gives for small $s > 0$ that $L(c_{v,s}) < 2R_0 = L(\gamma)$. Thus there are curves as close as we like to γ with both endpoints on ∂M and with length $< 2R_0$. This contradicts part 3 of Proposition 2.2 and completes the proof of Theorem 4.4 in the Euclidean case. The mean curvature H of ∂M is given by $(n-1)H = \operatorname{trace} I\!I$. Thus if in (5.1) we let v vary over an orthonormal basis $e_1, \dots, e_{n-1}$ and sum:

$$\sum_{i=1}^{n-1} \frac{d^2}{ds^2} L(c_{e_i,s})\bigg|_{s=0} = -\operatorname{trace}(I\!I_1) - \operatorname{trace}(I\!I_2) = -(n-1)(H_1 + H_2).$$

If the mean curvature of ∂M is strictly positive we again get a contradiction to part 3 of Proposition 2.2 which implies all second variations $d^2/ds^2 L(c_{e_i,s})|_{s=0}$ most be non-negative. While this argument does not cover the general case of H non-negative it does avoid use of elliptic theory.

Let $\lambda_1, \dots, \lambda_{n-1}$ be the principal curvatures of ∂M with respect to $\mathbf{n}$ and for $r \geq 0$ define $F_r \colon \partial M \to \mathbf{R}^n$ by $F_r(x) := x + r\mathbf{n}(x)$. Then F_r parameterizes the parallel hypersurface to ∂M at a distance of r from ∂M. Near a point x of ∂M the function F_r is an immersion (i.e. the derivative of F_r is nonsingular) near x for all $r < \operatorname{focal}_{\partial M}(x)$. A calculation shows the mean curvature $H(r)$ of the image of F_r is $(n-1)H(r) = \sum_1^{n-1} \lambda_i(1-r\lambda_i)^{-1}$. The derivative of this with respect to r is $(n-1)H'(r) = \sum_1^{n-1} \lambda_i^2(1-r\lambda_i)^{-2}$. Thus $H(r)$ is a strictly increasing function of r unless $\lambda_1 = \lambda_2 = \cdots = \lambda_{n-1} = 0$. Assume, toward a contradiction, that $H \geq$ on ∂M, but that $R_0 := \operatorname{Roll}(M) < \operatorname{Focal}(M)$. We use the notation of the last couple of paragraphs above (i.e. γ a segment of the x_n axis, U_i the graph of f_i etc.). Let $U_1[2R_0] := F_{2R_0}[U_1]$ be the parallel hypersurface to U_1 at a distance of $2R_0$. Near $\gamma(2R_0) = (0, 2R_0)$ it is possible to write $U_1[2R_0]$ as a graph of a function $x_n = f_3(x)$ with $x \in \mathbf{R}^{n-1}$ close to the origin of $\mathbf{R}^{n-1}$. By Proposition 2.2 points of U_2 are at a distance $\geq 2R_0$ from U_1 which implies $f_3(x) \leq f_2(x)$ for all x in a neighborhood $0 \in \mathbf{R}^{n-1}$. But the mean curvature of $U_1[2R_0]$ is non-negative with respect to the direction of increasing x_n and the mean curvature of U_2 in non-positive with respect to the direction of increasing x_n. The equation for mean curvature is quasi-linear elliptic and thus these curvature inequalities and along with $f_3 \leq f_2$ and $f_3(0) = f_2(0)$ implies $f_3 = f_2$ in a neighborhood of $0 \in \mathbf{R}^{n-1}$ (cf [27, Lemma 1 p798] or [10, Thm 1]) Whence the mean curvature of $U_3[2R_0]$ vanishes. By the remarks above this can only happen if all the principal curvatures of U_1 vanish, that is only if U_1 is a subset of $\mathbf{R}^{n-1} \times \{0\}$. Thus $U_2 = U_1[2R_0]$ is a subset of $\{x_n = 2R_0\}$. A continuation argument now shows ∂M must contain all of $\mathbf{R}^{n-1} \times \{0\}$ which is impossible as ∂M is compact. This completes the proof of Theorem 4.3 in the Euclidean case. □

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