KUIPER'S THEOREM ON CONFORMALLY FLAT MANIFOLDS

RALPH HOWARD

DEPARTMENT OF MATHEMATICS
UNIVERSITY OF SOUTH CAROLINA
COLUMBIA, S.C. 29208, USA
HOWARD@MATH.SC.EDU

1. Introduction

These are notes to that show how to modify the proof given by Kuiper [2] that a compact simply connected conformally flat manifold is conformally diffeomorphic to a sphere under less restrictive smoothness conditions (Kuiper works with metrics of class C^3). The proof mostly follows the original proof of Kuiper other than we use a covering space argument rather than his monodromy argument and we restrict ourselves to Riemannian metrics while Kuiper works with conformally flat semi-Riemannian manifolds. What allows us to extend Kuiper's proof is a theorem of Gehring [1] which shows that the theorem of Louisville on conformal maps between Euclidean space of dimension three or more holds for C^1 maps.

2. Regularity of Conformal Maps.

By a C^1 conformally flat manifold we mean a Riemannian manifold (M, g) so that M is of class C^1, the metric is of class C^0 and every point has a C^1 coordinate system $x^1, \dots, x^n$ so that in this coordinate path the metric has the form $g = \lambda^2((dx^1)^2 + \cdots + (dx^n)^2)$. Or what is the same thing that M has a cover by open sets $\{U_\alpha\}$ which are the domain of C^1 diffeomorphisms $\varphi_\alpha\colon U_\alpha \to \varphi_\alpha[U_\alpha] \subset \mathbf{R}^n$ onto open sets so that the transition functions $\Phi_{\alpha,\beta} := \varphi_\alpha \circ \varphi_\beta^{-1}\big|_{\varphi_\beta[U_\alpha \cap U_\beta]}\colon \varphi_\beta[U_\alpha \cap U_\beta] \to \varphi_\alpha[U_\alpha \cap U_\beta]$ are of class C^1 and if g_0 is the flat metric on $\mathbf{R}^n$ then $\varphi_\alpha^* g_0 = \lambda_\alpha^2 g$ for some positive continuous function λ_α defined on U_α.

As the sphere S^n with its standard metric is locally conformally flat we could just as well as taken the maps φ_α to have values in S^n rather than $\mathbf{R}^n$ and in what follows we will often do this.

We now recall the definition of a Möbius transformation of the sphere S^n. Let $\mathbf{R}^{n+2}_1$ be $\mathbf{R}^{n+2}$ with the Lorentzian inner product $g^n_1 = (dx^1)^2 + \cdots + (dx^{n+1})^2 - (dx^{n+2})^2$ and let $O^+(n+2, 1)$ be the group of linear maps that preserve both the inner product and the direction of the "time axis"

Date: July 1996.

x^{n+2}. Embed S^n in $\mathbf{R}^{n+2}_1$ as $S^n := \{(x,1) \in \mathbf{R}^{n+1} \times \mathbf{R} : \|x\| = 1\}$ where $\|x\|^2 = (x^1)^2 + \cdots + (x^{n+1})^2$ is the usual Euclidean norm on $\mathbf{R}^{n+1}$. Then the induced metric on S^n is the usual round metric. Each $a \in O^+(n+1,1)$ a map Ψ_a on S^n as follows. Let $a(x,1) = (a_1 x, a_2(x))$. Then as a preserves the time orientation the real number $a_2(x)$ is positive. Set

$$\Psi_a(x,1) = \left(\frac{a_1(x)}{a_2(x)}, 1\right).$$

Then $\Psi_a \Psi_b = \Psi_{ab}$ and each of the maps Ψ_a is conformal on S^n. (To see this note that $x \mapsto (a_1(x), a_2(x))$ is an isometry of S^n with $a[S^n]$ as a is an isometry of the ambient space $\mathbf{R}^{n+2}_1$ and $(a_1(x), a_2(x)) \mapsto a_2(x)^{-1}(a_1(x), a_2(x) = \Psi_a(x,1)$ is then a conformal map.) The maps Ψ_a with $a \in O^+(n+2,1)$ are ***Möbius transformations***. The collection of all Möbius transformations is then a group of conformal transformations acting on S^n called the ***Möbius group***. The proof of Kuiper's theorem is based on

Theorem 2.1 (Louisville-Gehring)**.** *Assume $n \geq 3$ and that $U \subset S^n$ is open connected set. Then any C^1 conformal map $\varphi \colon U \to S^n$ is the restriction of a Möbous transformation.*

Proof. This is Theorem 16 on page 389 of Gehring's paper [1]. Louisville had proven the result under the assumption the map is of class C^4. Gehring reduces the more general result to this case by proving a regularity theorem which implies that a C^1 conformal map is real analytic. In fact the result of Gehring is even more general than the C^1 result as he has a definition of what it means for a continuous map to be conformal and then proves his regularity theorem for maps that conformal in this sense except on a set of finite $n-1$ dimensional measure. Also Gehring works in $\mathbf{R}^n$ instead of S^n, but the S^n results follows easily by use of stereographic projection.

Here we give a different proof that, while not quite rigorous, uses the machinery of Riemannian geometry rather than that of quasiconformal maps. We first recall some facts about the conformal Laplacian. Let g and $\overline{g}$ be two conformal metrics on manifold M, say $\overline{g} = \lambda^2 g$. Let S be the scalar curvature and Δ the Laplacian of g and $\overline{S}$ and $\overline{\Delta}$ the scalar curvature of $\overline{g}$. Then for any smooth function u on M

$$\left(\overline{\Delta} - \frac{(n-2)\overline{S}}{4n-4}\right)(\lambda^{\frac{2-n}{2}} u) = \lambda^{-\frac{n+2}{2}} \left(\Delta - \frac{(n-2)S}{4n-4}\right) u.$$

As a special case let M be a connected open set in $\mathbf{R}^n$ and let $\varphi \colon U \to \mathbf{R}^n$ be conformal, so that for some positive function $\lambda \colon U \to \mathbf{R}$ there holds $\varphi^* g = \lambda^2 g$. For the time being we don't worry about the smoothness of φ and just assume that it has all the derivatives we need. Let $\overline{g} := \varphi^* g = \lambda^2 g$. Then $\overline{g}$ is the pullback of the flat metric and so it is also flat. Therefore both g and $\overline{g}$ have zero scalar curvature. Therefore in our case the last equation becomes

$$\overline{\Delta}(\lambda^{\frac{2-n}{2}} u) = \lambda^{-\frac{n+2}{2}} \Delta u. \tag{2.1}$$

Now let $u := \lambda^{\frac{n-2}{2}}$ so that $\lambda^{\frac{2-n}{2}}u \equiv 1$. Then equation (2.1) and $\overline{\Delta}1 = 0$ yields $\Delta\lambda^{\frac{n-2}{2}} = 0$ so that by Weyl's lemma $\lambda^{\frac{n-2}{2}}$ and therefore also λ is real analytic. Write $\varphi = (\varphi^1, \dots, \varphi^n)$ so that $\varphi^i = \varphi^* x^i$. Then as the coordinate functions x^i are harmonic with respect to the flat metric the functions φ^i are harmonic with respect to the metric $\overline{g}$. Therefore $\overline{\Delta}\varphi^i = 0$. So if we let $u = \lambda^{\frac{n-2}{2}}\varphi^i$ in (2.1) we find $\Delta(\lambda^{\frac{n-2}{2}}\varphi^i) = 0$ so that φ^i is also real analytic. Whence φ is real analytic.

To make this all rigorous in the case φ is only C^1 it is enough to show that all all the statements can be interpreted in the weak sense and that they still hold in this sense when φ is only C^1. While I have not done this my deep faith in elliptic technology makes me believe it works. □

Corollary 2.2. *Any C^1 conformally flat manifold (M, g) has a natural real analytic structure and there is a C^∞ metric in the conformal class of g.*

Proof. Let $\{U_\alpha\}$ an open cover of M so that there are C^1 conformal maps $\varphi_\alpha \colon U_\alpha \to \varphi_\alpha[U_\alpha] \subset S^n$ as in the definition of a C^1 conformal manifold above and let $\Phi_{\alpha,\beta} := \varphi_\alpha \circ \varphi_\beta^{-1}\big|_{\varphi_\beta[U_\alpha \cap U_\beta]} \colon \varphi_\beta[U_\alpha \cap U_\beta] \to \varphi_\alpha[U_\alpha \cap U_\beta]$ be the corresponding transition functions. Then by the theorem of Louisville-Gehring the restriction of $\Phi_{\alpha,\beta}$ to any connected component of $\varphi_\beta[U_\alpha \cap U_\beta]$ is the restriction of a Möbius transformation and therefore $\Phi_{\alpha,\beta}$ is real analytic. This gives a real analytic atlas on M.

Let g_0 be the standard metric on S^n. The metric $g_\alpha := \varphi_\alpha^* g_0$ on U_α is real analytic with respect to the real analytic structure we have defined on M and is also conformal to the metric g on U_α on U_α. These metrics can be pieced together by use of a partition of unity to give a smooth metric in the conformal class of g. □

3. Kuiper's Theorem

Theorem 3.1 (Kuiper [2])**.** *Let (M^n, g) be a simply connected conformally flat manifold of class C^1. Then there is a conformal immersion $f \colon M \to S^n$. If M is compact then this map is a conformal diffeomorphism of M with S^n.*

Proof. Let $\mathcal{U} = \{U_\alpha\}$ be an open cover of M by connected open sets so that for each α there is a C^1 injective conformal map $\varphi_\alpha \colon U_\alpha \to S^n$. We now claim that we can find anther open cover $\mathcal{V}$ and for each $V \in \mathcal{V}$ an injective conformal map $\psi_V \colon V \to S^n$ so that $\mathcal{V}$ is countable, each $V \in \mathcal{V}$ and each intersection $V_1 \cap V_2$ with $V_1, V_2 \in \mathcal{V}$ is connected. To see this we use that M is paracompact so that we can express $M = \cup_{i=1}^\infty K_i$ as a countable union of compact sets K_i and so that this union is locally finite. Now choose a complete Riemannian metric h on M (which need not be related to the conformal structure). For each i let $\delta_{i,1}$ be the Lebesgue number of the cover on K_i. That is if $r \le \delta_{i,1}$ and $x \in K_i$ then the open ball $B(x, r)$ is a subset of some member U_α of $\mathcal{U}$. Let $\delta_{i,2}$ be the convexity radius of K_i in M, that is if $x \in K_i$ and $r \le \delta_{i,2}$ then the ball $B(x, r)$ is convex in the sense that any two points of $B(x,r)$ can be joined by a unique minimizing segment and this segment is a subset of $B(x,r)$. Let $\delta_i := \min(\delta_{i,1}, \delta_{i,2})$. Then $\{B(x,\delta_i) : x \in K_i\}$ is an open cover of K_i and therefore it has a finite sub-cover $\mathcal{V}_i$. For each $V \in V_i$ by construction there is a $U_\alpha \in \mathcal{U}$ so that $V \subseteq U_\alpha$. Set $\psi_V = \varphi_\alpha\big|_V$. Finally set $\mathcal{V} := \cup_{i=1}^\infty \mathcal{V}_i$. This is a countable union of finite sets and thus countable and the connectivity of the intersection $V_1 \cap V_2$ follows from the convexity of V_1 and V_2.

Let $V_1, V_2 \in \mathcal{V}$ and $V_1 \cap V_2 \neq \emptyset$ then by the theorem of Louisville-Gehring the transition function $\Psi_{V_1,V_2} := \psi_{V_2} \circ \psi_{V_1}\big|_{\psi_{V_1}[V_1 \cap V_2]} \colon \psi_{V_1}[V_1 \cap V_2] \to \psi_{V_2}[V_1 \cap V_2]$ is the restriction of a Möbius transformation. This implies there is a unique Möbius transformation a so that ψ_{V_1} and $a \circ \psi_{V_2}$ agree on the set $V_1 \cap V_2$. We then say that $a \circ \psi_{V_2}$ is the ***analytic continuation*** of ψ_{V_1} into V_2. More generally given a chain of open $V_0, V_1, \dots, V_k \in \mathcal{V}$ (by a chain we mean that $V_i \cap V_{i+1} \neq \emptyset$ for $i = 0, \dots, k-1$) we can repeat this and get a unique analytic continuation of ψ_{V_0} to the set V_k along the given chain. We now want to show that when M is simply connected this analytic continuation is independent of the chain connecting V_0 and V_k.

Toward this end we fix a $V_0 \in \mathcal{V}$ to use as a starting point for our construction. For any $V \in \mathcal{V}$ let $\mathcal{C}_V$ be the collection of all analytic continuations of ψ_{V_0} to V any chain $V_0, \dots, V_k = V$ connecting V_0 and V. Define a submanifold $G \subset M \times S^n$ by

$$G := \bigcup_{V \in \mathcal{V}} \{(x, \psi(x)) : x \in V, \psi \in \mathcal{C}_V\},$$

where the manifold structure is defined so that the projection onto M is a local diffeomorphism. If $V_0, \dots, V_k = V$ is a chain and $\psi_{V_0} =: \varphi_0, \varphi_1, \dots, \varphi_k$ are the maps obtained by analytically continuing ψ_{V_0} along the chain. Then $\cup_{i=0}^k \{(x, \varphi_i(x)) : x \in V_i\}$ is a connected subset of G. But from the definition G this means that every point of G is in the same connected component as $(x, \psi_0(x))$ for $x \in V_0$ thus G is connected. Let $\pi \colon G \to M$ be the restriction of projection onto the first factor. Then for each $V \in \mathcal{V}$ the set $\pi^{-1}[V]$ is the disjoint[1] union of the sets $\{(x, \psi(x)) : x \in V\}$ where ψ varies over $\mathcal{C}_V$ and the restriction of π to any of one of the sets $\{(x, \psi(x)) : x \in V\}$ is a diffeomorphism with V. Thus each of the sets V is evenly covered by the map $\pi \colon G \to M$. Thus $\pi \colon G \to M$ is a covering map. As we are assuming that M is simply connected and we have shown G is connected this means that $\pi \colon G \to M$ is a diffeomorphism. Therefore G is the graph of a function $f \colon M \to S^n$ which is easily seen to be a conformal immersion.

Finally assume that M is compact. As the map $f \colon M \to S^n$ is a conformal immersion the image of f is an open subset of S^n. But M compact implies it is also closed. As S^n is connected the map f is surjective. For any point $x \in M$ there is a neighborhood U of x so that f is injective on U. By the

[1]Strictly speaking to insure that this union is disjoint we should not use the submanifold topology on G, but rather the topology in inherits as subset set of the sheaf of germs of smooth S^n valued functions.

compactness of M we can then cover M by open sets $U_1, \ldots, U_k$ so that f is injective on each U_i. This implies any point $y \in S^n$ has at most k preimages. So let $f^{-1}[y] := \{x_1, \ldots, x_l\}$. Let N_i be a open neighborhood of x_i so that $N_1, \ldots, N_l$ are pairwise disjoint and so that f is injective on each N_i. Then $U := \cap_{i=1}^{1} f[N_i]$ is an open neighborhood of y and $f^{-1}[U] := \cup_{i=1}^{l}(f^{-1}[U] \cap N_i)$ and $f\big|_{f^{-1}[U]\cap N_i} : f^{-1}[U] \cap N_i \to U$ is a diffeomorphism for all $i = 1, \ldots, l$. Therefore f is a covering map and as S^n is simply connected this means it is a diffeomorphism. □

References

1. F. W. Gehring, *Rings and quasiconformal mappings in space*, Trans. Amer. Math. Soc. **103** (1962), 353–391.
2. N. H. Kuiper, *On conformally flat spaces in the large*, Ann. Math. **50** (1949), 916–924.