A COUNTEREXAMPLE TO ONE OF MY FAVORITE CONJECTURES.

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Notes from a talk given in the analysis seminar at the University of South Carolina 3 November, 2006.

1. Introduction.

Let M be a smooth compact manifold and let $C^\infty(M)$ be the Fréchet space of all real (or if it is convenient complex) valued functions on M. We can define the topology on $C^\infty(M)$ by either of two sets of semi-norms on $C^\infty(M)$. Fix a Riemannian metric on M, and define $p_k\colon C^\infty(M) \to [0,\infty)$ by

$$p_k(u) = \max |\nabla^k u|$$

where ∇^k is the k-th covariant derivative of u with respect to the connection on M defined by the metric. Then the family of semi-norms $\{p_k\}_{k=0}^\infty$ defines the topology on $C^\infty(M)$ and this topology is independent of the choice of the Riemannian metric. From now on we will assume that our manifolds are equipped with a smooth Riemannian metric. There is another set of semi-morns that defines the same topology and may be even more natural. Let Δ be the Laplacian on M with the sign chosen so that Δ is a positive semi-definite operator. For $k \geq 0$ set

$$q_k(u) = \|(I+\Delta)^{k/2}u\|_{L^2(M)}.$$

Then, as follows from the Sobolev inequality, the semi-norms $\{q_k\}_{k=0}^\infty$ define the same topology on $C^\infty(M)$.

Definition 1. If M and N are smooth compact Riemannian manifolds. Then a linear operator $L\colon C^\infty(M) \to C^\infty(N)$ is ***of finite order*** iff there is an integer ℓ, the ***order*** of L, so that for all k, there is constant C_k such that

$$q_k(Lu) \leq C_k q_{k+\ell}(u).$$

(This can also be defined in terms of the semi-norms p_k, but the value of ℓ may change.)

About 15 ten years ago I made the

Conjecture 1. *If M and N are compact Riemannian manifolds with* $\dim M > \dim N$*, then any linear map $L\colon C^\infty(M) \to C^\infty(N)$ with finite order has an infinite dimensional kernel and thus is not injective.* □

The motivation for this is as follows. Let $\mathbf{Gr}_k(\mathbf{R}^n)$ be the Grassmannian of all linear subspaces of $\mathbf{R}^n$. This is a Riemannian manifold of dimension $k(n-k)$. For

Date: October, 2006.

if $j, k \in \{1, 2, \ldots, n-1\}$ define the ***Radon transform*** by $R_{j,k}\colon C^\infty(\mathbf{Gr}_j(\mathbf{R}^n)) \to C^\infty(\mathbf{Gr}_k(\mathbf{R}^n))$ by

$$R_{j,k}f(P) = \begin{cases} \displaystyle\int_{\{L \in G_k(\mathbf{R}^n) : L \subseteq P\}} f(L)\, dL, & \text{if } j \le k; \\ \displaystyle\int_{\{U \in G_k(\mathbf{R}^n) : U \supset P\}} f(U)\, dU, & \text{if } j > k. \end{cases}$$

It is well known in the cases where $\dim \mathbf{Gr}_j(\mathbf{R}^n) > \dim \mathbf{Gr}_k(\mathbf{R}^n)$ that kernel$(R_{j.k})$ is infinite dimensional. This is one of many examples, all of which have finite order, (Radon transforms between complex or quaternionic Grassmannians, cosine transforms and generalizations between real Grassmannians etc.) where Conjecture 1 holds. Thus, if it were true, the conjecture would explain a lot of non-injectivity results in integral geometry.

Our main result is

Proposition 1. *Given any two compact Riemannian manifolds M and N (with no restrictions on the dimensions) there is an injective linear $L\colon C^\infty(M) \to C^\infty(N)$ of order 0.*

And choice of M and N with $\dim M > \dim N$ gives a counterexample to the conjecture.

2. Proof of Proposition 1.

Let M be a compact Riemannian manifold and $0 = \lambda_0(M) < \lambda_1(M) \le \lambda_2(M) \le \lambda_3(M) \le \cdots$ the eigenvalues of Δ on M. Then there is an orthonormal basis $\{\varphi_k^M\}_{k=0}^\infty$ of $L^2(M)$ such that $\Delta \varphi_k^M = \lambda_k(M)$. If $n = \dim M$, then (cf. [1, p. 155]) ***Weyl's asymptotic formula***

$$(\lambda_k(M))^{n/2} \sim \frac{(2\pi)^n k}{\omega_m} \mathrm{Vol}(M)$$

holds (where ω_m is the volume of the unit ball in $\mathbf{R}^n$). This gives the order of growth of $\lambda_k(M)$ as

$$\lambda_k(M) \sim (2\pi)^2 \left(\frac{\mathrm{Vol}(M)}{\omega_m} \right)^{2/n} k^{2/n} = C_M k^{2/n}. \tag{1}$$

If $u \in C^\infty(M)$ has expansion $u = \sum_{k=0}^\infty u_k \varphi_k^M$, then $(I + \Delta)\varphi_k^M = (1 + \lambda_k(M))\varphi_k^M$ and thus

$$\begin{aligned} q_m(u) &= \|(I+\Delta)^{m/2} u\|_{L^2(M)} \\ &= \Big\| \sum_{k=0}^\infty (1 + \lambda_k(M))^{m/2} u_k \varphi_k^M \Big\|_{L^2(M)} \\ &= \Big(\sum_{k=0}^\infty (1 + \lambda_k(M))^m |u_k|^2 \Big)^{1/2}. \end{aligned}$$

Thus, in light of (1), $q_m(u) < \infty$ for all m if and only if the sequence of Fourier coefficients $\{u_k\}_{k=0}^\infty$ has faster than polynomial decrease.

Let $\{\beta_k\}_{k=0}^{\infty}$ be a sequence of positive real numbers with faster than polynomial decay, say $\beta_k = e^{-k}$. Let N be any other compact Riemannian manifold and define $K\colon N \times M \to \mathbf{R}$ by

$$K(y,x) := \sum_{k=0}^{\infty} \beta_k \varphi_k^N(y)\varphi_k^M(x).$$

It then follows from the Weyl formula (1) and estimates due to P. Li (cf. [1, Thm 8 p. 102]) that this series converges uniform and therefore K is continuous. (With some more work it can be shown that K is C^∞ on $N \times M$.) If L is the integral operator $L\colon C^\infty(M) \to C^\infty(N)$

$$Lu = \int_M K(y,x)u(x)\,dx$$

then

$$L\varphi_k^M = \beta_k \varphi_k^N.$$

(We could also just define L by this formula and ignore K, but it is nice to know that L is represented by a smooth kernel.) If $u = \sum_{k=0}^{\infty} u_k \varphi_k^M$ then

$$Lu = \sum_{k=0}^{\infty} \beta_k u_k \varphi_k^N$$

which shows that L is injective. Also

$$q_m(Lu) = \Big(\sum_{k=0}^{\infty}(1+\lambda_k(N))^m \beta_k^2 |u_k|^2\Big)^{1/2}. \tag{2}$$

Because of the faster than polynomial decay of β_k and the estimates (1)

$$\lim_{k\to\infty} \frac{(1+\lambda_k(N))^m\beta_k^2}{(1+\lambda_k(M))^m} = 0$$

and therefore there is a constant C_m such that

$$(1+\lambda_k(N))^m\beta_k^2 \le C_m^2(1+\lambda_k(M))^m$$

for all k. Using this in (2) gives

$$q_m(u) \le C_m\Big(\sum_{k=0}^{\infty}(1+\lambda_k(M))^m|u_k|^2\Big)^{1/2} = C_m q_m(u).$$

Therefore L has order 0. This completes the proof.

Remark 1. The operator L, being an integral operator with smooth kennel, is even nicer than just being of finite order. It is infinitely smoothing and thus can be considered of order $-\infty$.

References

[1] I. Chavel, *Eigenvalues in Riemannian geometry*, Academic Press Inc., Orlando, Fla., 1984, Including a chapter by Burton Randol, With an appendix by Jozef Dodziuk. MR 86g:58140

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