Mathematics 700 Final

1. (10 points) State and prove the rank plus nullity theorem.

2. (20 points) Let $\mathcal{P}_3$ be the vector space of real polynomials of degree ≤ 3. Define a linear map $T : \mathcal{P}_3 \to \mathcal{P}_3$ by

$$(Tf)(x) = x(f(x+1) - f(x))$$

   (a) Find the null space of T.

   (b) Find the rational canonical form of T.

   (c) Find the Jordan canonical form of T.

   (d) Find the range of T.

3. (10 points) Let $\mathbf{R}^{3*}$ be the dual space to $\mathbf{R}^3$. Then find the basis of $\mathbf{R}^{3*}$ dual to the basis $(1,0,0)$, $(1,1,0)$, $(0,1,1)$ of $\mathbf{R}^3$.

4. (10 points) Let V be a finite dimensional vector space over the a field $\mathbf{F}$ and let $T \in L(V,V)$. Then state what the primary decomposition of V with respect to T is.

5. (10 points) Let $P \in L(V,V)$ where V is a finite dimensional vector space. Assume that $P^2 = P$ and show that $\operatorname{rank}(P) = \operatorname{trace}(P)$.

6. (10 points) Let $A = \begin{bmatrix} 0 & -1 \\ 4 & 0 \end{bmatrix}$. Then find a basis of $\mathbf{R}^2$ that makes A diagonal or show that no such basis exists.

7. (10 points) Let V be a finite dimensional vector space and S a lienar opeartor on V. Let $W \subseteq V$ be a subspace that is invariant under S. Choose a basis $w_1, \dots, w_k$ of W and extend it to a basis $w_1, \dots, w_k, v_{k+1}, \dots, v_n$ of V. Let A be the matrix of V in this basis. Then what can you say about the form of A?

8. (5 points) Let $f(x) = x^3 + ax^2 + bx + c$. Then find a 3×3 matrix that has $f(x)$ as a minimal polynomial.

9. (5 points) Give an of a 5×5 matrix A with minimal polynomial $(x-1)^2(x-2)(x-3)$ and $\det A = 12$.

10. (5 points) Let A be a 2×2 matrix with $\operatorname{trace} A = 0$. Then show $A^2 = -\det(A)I$.

11. (5 points) Let $U, V \subset \mathbf{R}^7$ be subspaces with $\dim U = 5$ and $\dim V = 6$. Then what can you say about $\dim(U \cap V)$?.