Linear Algebra Questions from the Admission to Candidacy Exam

The following is a more or less complete list of the linear algebra questions that have appeared on the admission to candidacy exam for the last ten years. I have reworded some of them a little and have omitted some questions that have been repeated. Following the exam questions are some problems that I think are important.

January 1984

1. Let V be a finite-dimensional vector space and let T be a linear operator on V. Suppose that T commutes with every diagonalizable linear operator on V. Prove that T is a scalar multiple of the identity operator.

2. Let V and W be vector spaces and let T be a linear operator from V into W. Suppose that V is finite-dimensional. Prove $\operatorname{rank}(T) + \operatorname{nullity}(T) = \dim V$.

3. Let A and B be $n \times n$ matrices over a field F

   (a) Prove that if A or B is nonsingular, then AB is similar to BA.

   (b) Show that there exist matrices A and B so that AB is *not* similar to BA.

   (c) What can you deduce about the eigenvalues of AB and BA. Prove your answer.

4. Let $A = \begin{pmatrix} D & E \\ F & G \end{pmatrix}$, where D and G are $n \times n$ matrices. If $DF = FD$ prove that $\det A = \det(DG - FE)$.

5. If F is a field, prove that every ideal in $F[x]$ is principle.

August 1984

1. Let V be a finite dimensional vector space. Can V have three distinct proper subspaces W_0, W_1 and W_2 such that $W_0 \subseteq W_1$, $W_0 + W_2 = V$, and $W_1 \cap W_2 = \{0\}$?

2. Let n be a positive integer. Define

$$G = \{A : \ A \text{ is an } n \times n \text{ matrix with only integer entries and } \det A \in \{-1, +1\}\},$$
$$H = \{A : \ A \text{ is an invertible } n \times n \text{ matrix and both } A \text{ and } A^{-1} \text{ have only integer entries}\}.$$

   Prove $G = H$.

3. Let V be the vector space over $\mathbf{R}$ of all $n \times n$ matrices with entries from $\mathbf{R}$.

   (a) Prove that $\{I, A, A^2, \ldots, A^n\}$ is linearly dependent for all $A \in V$.

   (b) Let $A \in V$. Prove that A is invertible if and only if I belongs to the span of $\{A, A^2, \ldots, A^n\}$.

4. Is every $n \times n$ matrix over the field of complex numbers similar to a matrix of the form $D + N$ where D is a diagonal matrix, $N^{n-1} = 0$, and $DN = ND$. Why?

January 1985

1. (a) Let V and W be vector spaces and let T be a linear operator from V into W. Suppose that V is finite-dimensional. Prove $\operatorname{rank}(T) + \operatorname{nullity}(T) = \dim V$.
   (b) Let $T \in L(V, V)$, where V is a finite dimensional vector space. (For a linear operator S denote by $\mathcal{N}(S)$ the null space and by $\mathcal{R}(S)$ the range of S.)
      i. Prove there is a least natural number k such that $\mathcal{N}(T^k) = \mathcal{N}(T^{k+1}) = \mathcal{N}(T^{k+2}) \cdots$ Use this k in the rest to this problem.
      ii. Prove that $\mathcal{R}(T^k) = \mathcal{R}(T^{k+1}) = \mathcal{R}(T^{k+2}) \cdots$
      iii. Prove that $\mathcal{N}(T^k) \cap \mathcal{R}(T^k) = \{0\}$.
      iv. Prove that for each $\alpha \in V$ there is exactly one vector in $\alpha_1 \in \mathcal{N}(T^k)$ and exactly one vector $\alpha_2 \in \mathcal{R}(T^k)$ such that $\alpha = \alpha_1 + \alpha_2$.

2. Let F be a field of characteristic 0 and let

$$W = \left\{ A = [a_{ij}] \in F^{n\times n} : \operatorname{tr}(A) = \sum_{i=1}^{n} a_{ii} = 0 \right\}.$$

For $i, j = 1, \ldots, n$ with $i \neq j$, let E_{ij} be the $n \times n$ matrix with (i, j)-th entry 1 and all the remaining entries 0. For $i = 2, \ldots, n$ let E_i be the $n \times n$ mat-ix with $(1, 1)$ entry -1, (i, i)-th entry $+1$, and all remaining entries 0. Let

$$S = \{E_{ij} : i, j = 1, \ldots, n \text{ and } i \neq j\} \cup \{E_i : i = 2, \ldots, n\}.$$

[Note: You can assume, without proof, that S is a linearly independent subset of $F^{n\times n}$.]

   (a) Prove that W is a subspace of $F^{n\times n}$ and that $W = \operatorname{span}(S)$. What is the dimension of W?
   (b) Suppose that f is a linear functional on $F^{n\times n}$ such that
      i. $f(AB) = f(BA)$, for all $A, B \in F^{n\times n}$.
      ii. $f(I) = n$, where I is the identity matrix in $F^{n\times n}$.

Prove that $f(A) = \operatorname{tr}(A)$ for all $A \in F^{n\times n}$.

August 1985

1. Let V be a vector space over $\mathbf{C}$. Suppose that f and g are linear functionals on V such that the functional

$$h(\alpha) = f(\alpha)g(\alpha) \quad \text{for all} \quad \alpha \in V$$

is linear. Show that either $f = 0$ or $g = 0$.

2. Let C be a 2×2 matrix over a field F. Prove: There exists matrices $C = AB - BA$ if and only if $\operatorname{tr}(C) = 0$.

3. Prove that if A and B are $n \times n$ matrices from $\mathbf{C}$ and $AB = BA$, then A and B have a common eigenvector.

January 1986

1. Let F be a field and let V be a finite dimensional vector space over F. Let $T \in L(V,V)$. If c is an eigenvalue of T, then prove there is a nonzero linear functional f in $L(V,F)$ such that $T^*f = cf$. (Recall that $T^*f = fT$ by definition.)

2. Let F be a field, $n \geq 2$ be an integer, and let V be the vector space of $n \times n$ matrices over $\mathbf{F}$. Let A be a fixed element of V and let $T \in L(V,V)$ be defined by $T(B) = AB$.

   (a) Prove that T and A have the same minimal polynomial.

   (b) If A is diagonalizable, prove, or disprove by counterexample, that T is diagonalizable.

   (c) Do A and T have the same characteristic polynomial? Why or why not?

3. Let M and N be 6×6 matrices over $\mathbf{C}$, both having minimal polynomial x^3.

   (a) Prove that M and N are similar if and only if they have the same rank.

   (b) Give a counterexample to show that the statement is false if 6 is replaced by 7.

August 1986

1. Give an example of two 4×4 matrices that are not similar but that have the same minimal polynomial.

2. Let $(a_1, a_2, \ldots, a_n)$ be a nonzero vector in the real n-dimensional space $\mathbf{R}^n$ and let P be the hyperplane
$$\left\{ (x_1, x_2, \ldots, x_n) \in \mathbf{R}^n : \sum_{i=1}^{n} a_i x_i = 0 \right\}.$$
Find the matrix that gives the reflection across P.

January 1987

1. Let V and W be finite-dimensional vector spaces and let $T : V \to W$ be a linear transformation. Prove that that exists a basis $\mathcal{A}$ of V and a basis $\mathcal{B}$ of W so that the matrix $[T]_{\mathcal{A},B}$ has the block from $\begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix}$.

2. Let V be a finite-dimensional vector space and let T be a diagonalizable linear operator on V. Prove that if W is a T invariant subspace then the restriction of T to W is also diagonalizable.

3. Let T be a linear operator on a finite-dimensional vector. Show that if T has no cyclic vector then, then there exists an operator U on V that commutes with T but is *not* a polynomial in T.

August 1987

1. Exhibit two real matrices with no real eigenvalues which have the same characteristic polynomial and the same minimal polynomial but are not similar.

2. Let V be a vector space, not necessarily finite-dimensional. Can V have three distinct proper subspaces A, B, and C, such that $A \subset B$, $A + C = V$, and $B \cap C = \{0\}$?

3. Compute the minimal and characteristic polynomials of the following matrix. Is it diagonalizable?

$$\begin{bmatrix} 5 & -2 & 0 & 0 \\ 6 & -2 & 0 & 0 \\ 0 & 0 & 0 & 6 \\ 0 & 0 & 1 & -1 \end{bmatrix}$$

August 1988

1. (a) Prove that if A and B are linear transformations on an n-dimensional vector space with $AB = 0$, then $r(A) + r(B) \leq n$ where $r(\cdot)$ denotes rank.

   (b) For each linear transformation A on an n-dimensional vector space, prove that there exists a linear transformation B such that $AB = 0$ and $r(A) + r(B) = n$.

2. (a) Prove that if A is a linear transformation such that $A^2(I - A) = A(I - A)^2 = 0$, then A is a projection.

   (b) Find a non-zero linear transformation so that $A^2(I - A) = 0$ but A is *not* a projection.

3. If S is an m-dimensional vector space of an n-dimensional vector space V, prove that S°, the annilihilator of S, is an $(n - m)$-dimensional subspace of V^*.

4. Let A be the 4×4 real matrix

$$A = \begin{bmatrix} 1 & 1 & 0 & 0 \\ -1 & -1 & 0 & 0 \\ -2 & -2 & 2 & 1 \\ 1 & 1 & -1 & 0 \end{bmatrix}$$

   (a) Determine the rational canonical form of A.

   (b) Determine the Jordan canonical form of A.

January 1989

1. Let T be the linear operator on $\mathbf{R}^3$ which is represented by

$$A = \begin{bmatrix} 1 & 1 & -1 \\ 1 & 1 & -1 \\ 1 & 0 & 0 \end{bmatrix}$$

   in the standard basis. Find matrices B and C which represent respectively, in the standard basis, a diagonalizable linear operator D and a nilpotent linear operator N such that $T = D + N$ and $DN = ND$.

2. Suppose T is a linear operator on $\mathbf{R}^5$ represented in some basis by a diagonal matrix with entries $-1, -1, 5, 5, 5$ on the main diagonal.

   (a) Explain why T can not have a cyclic vector.

   (b) Find k and the invariant factors $p_i = p_{\alpha_i}$ in the cyclic decomposition $\mathbf{R}^5 = \bigoplus_{i=1}^k Z(\alpha_i; T)$.

   (c) Write the rational canonical form for T.

3. Suppose that V in an n-dimensional vector space and T is a linear map on V of rank 1. Prove that T is nilpotent or diagonalizable.

August 1989

1. Let M denote an $m \times n$ matrix with entries in a field. Prove that

$$\begin{aligned} &\text{the maximum number of linearly independent rows of } M \\ = &\text{ the maximum number of linearly independent columns of } M \end{aligned}$$

   (Do not assume that $\operatorname{rank} M = \operatorname{rank} M^t$.)

2. Prove the Cayley-Hamilton Theorem, using only basic properties of determinants.

3. Let V be a finite-dimensional vector space. Prove there a linear operator T on V is invertible if and only if the constant term in the minimal polynomial for T is non-zero.

4. (a) Let $M = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 1 & -1 \end{bmatrix}$. Find a matrix T (with entries in $\mathbf{C}$) such that $T^{-1}MT$ is diagonal, or prove that such a matrix does not exist.

   (b) Find a matrix whose minimal polynomial is $x^2(x-1)^2$, whose characteristic polynomial is $x^4(x-1)^3$ and whose rank is 4.

5. Suppose A and B are linear operators on the same finite-dimensional vector space V. Prove that AB and BA have the same characteristic values.

6. Let M denote an $n \times n$ matrix with entries in a field F. Prove that there is an $n \times n$ matrix B with entries in F so that $\det(M + tB) \neq 0$ for every non-zero $t \in F$.

January 1990

1. Let W_1 and W_2 be subspaces of the finite dimensional vector space V. Record and prove a formula which relates $\dim W_1$, $\dim W_2$, $\dim(W_1 + W_2)$, $\dim(W_1 \cap W_2)$.

2. Let M be a symmetric $n \times n$ matrix with real number entries. Prove that there is an $n \times n$ matrix N with real entries such that $N^3 = M$.

3. TRUE OR FALSE. (If the statement is true, then prove it. If the statement is false, then give a counterexample.) If two nilpotent matrices have the same rank, the same minimal polynomial and the same characteristic polynomial, then they are similar.

August 1990

1. Suppose that $T : V \to W$ is a injective linear transformation over a field F. Prove that $T^* : W^* \to V^*$ is surjective. (Recall that $V^* = L(V, F)$ is the vector space of linear transformations from V to F.)

2. If M is the $n \times n$ matrix

$$M = \begin{bmatrix} x & a & a & \cdots & a \\ a & x & a & \cdots & a \\ a & a & x & \cdots & a \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a & a & a & \cdots & x \end{bmatrix}$$

then prove that $\det M = [x + (n-1)a](x-a)^{n-1}$.

3. Suppose that T is a linear operator on a finite dimensional vector space V over a field F. Prove that T has a cyclic vector if and only if

$$\{U \in L(V,V) : TU = UT\} = \{f(T) : f \in F[x]\}.$$

4. Let $T : \mathbf{R}^4 \to \mathbf{R}^4$ be given by

$$T(x_1, x_2, x_3, x_4) = (x_1 - x_4, x_1, -2x_2 - x_3 - 4x_4, 4x_2 + x_3)$$

   (a) Compute the characteristic polynomial of T.
   (b) Compute the minimal polynomial of T.
   (c) The vector space $\mathbf{R}^4$ is the direct sum of two proper T-invariant subspaces. Exhibit a basis for one of these subspaces.

January 1991

1. Let V, W, and Z be finite dimensional vector spaces over the field F and let $f : V \to W$ and $g : W \to Z$ be linear transformations. Prove that

$$\text{nullity}(g \circ f) \leq \text{nullity}(f) + \text{nullity}(g)$$

2. Prove that

$$\det \begin{bmatrix} A & 0 & 0 \\ B & C & D \\ 0 & 0 & E \end{bmatrix} = \det A \det B \det E$$

where A, B, C, D and E are all square matrices.

3. Let A and B be $n \times n$ matrices with entries on the field F such that $A^{n-1} \neq 0$, $B^{n-1} \neq 0$, and $A^n = B^n = 0$. Prove that A and B are similar, or show, with a counterexample, that A and B are not necessarily similar.

August 1991

1. Let A and B be $n \times n$ matrices with entries from $\mathbf{R}$. Suppose that A and B are similar over $\mathbf{C}$. Prove that they are similar over $\mathbf{R}$.

2. Let A be an $n \times n$ with entries from the field F. Suppose that $A^2 = A$. Prove that the rank of A is equal to the trace of A.

3. TRUE OR FALSE. (If the statement is true, then prove it. If the statement is false, then give a counterexample.) Let W be a vector space over a field F and let $\theta : V \to V'$ be a fixed surjective transformation. If $g : W \to V'$ is a linear transformation then there is linear transformation $h : W \to V$ such that $\theta \circ h = g$.

January 1992

1. Let V be a finite dimensional vector space and $A \in L(V,V)$.

   (a) Prove that there exists and integer k such that $\ker A^k = \ker A^{k+1} = \cdots$

(b) Prove that there exists an integer k such that $V = \ker A^k \oplus \operatorname{image} A^k$.

2. Let V be the vector space of $n \times n$ matrices over a field F, and let $T : V \to V^*$ be defined by $T(A)(B) = \operatorname{tr}(A^t B)$. Prove that T is an isomorphism.

3. Let A be an $n \times n$ matrix and $A^k = 0$ for some k. Show that $\det(A + I) = 1$.

4. Let V be a finite dimensional vector sauce over a field F, and T a linear operator on V. Suppose the minimal and characteristic polynomials of of T are the same power of an irreducible polynomial $p(x)$. Show that no non-trivial T-invariant subspace of V has a T-invariant complement.

August 1992

1. Let V be the vector space of all $n \times n$ matrices over a field F, and let B be a fixed $n \times n$ matrix that ti not of the form cI. Define a linear operator T on V by $T(A) = AB - BA$. Exhibit a not-zero element in the kernel of the transpose of T.

2. Let V be a finite dimensional vector space over a field F and suppose that S and T are triangulable operators on V. If $ST = TS$ prove that S and T have an eigenvector in common.

3. Let A be an $n \times n$ matrix over $\mathbf{C}$. If trace $A^i = 0$ for all $i > 0$, prove that A is nilpotent.

January 1993

1. Let V be a finite dimensional vector space over a field F, and let T be a linear operator on V so that $\operatorname{rank}(T) = \operatorname{rank}(T^2)$. Prove that V is the direct sum of the range of T and the null space of T.

2. Let V be the vector space of all $n \times n$ matrices over a field F, and suppose that A is in V. Define $T : V \to V$ by $T(AB) = AB$. Prove that A and B have the same characteristic values.

3. Let A and B be $n \times n$ over the complex numbers.
   (a) Show that AB and BA have the same characteristic values.
   (b) Are AB and BA similar matrices?

4. Let V be a finite dimensional vector space over a field of characteristic 0, and T be a linear operator on V so that $\operatorname{tr}(T^k) = 0$ for all $k \geq 1$, where $\operatorname{tr}(\cdot)$ denotes the trace function. Prove that T is a nilpotent linear map.

5. Let $A = [a_{ij}]$ be an $n \times n$ matrix over the field of complex numbers such that
$$|a_{ii}| > \sum_{j \neq i} |a_{ij}| \qquad \text{for} \qquad i = 1, \ldots, n.$$
Then show that $\det A \neq 0$. (det denotes the determinant.)

6. Let A be an $n \times n$ matrix, and let $\operatorname{adj}(A)$ denote the adjoint of A. Prove the rank of $\operatorname{adj}(A)$ is either 0, 1, or n.

August 1993

1. Let
$$A = \begin{bmatrix} 1 & 3 & 3 \\ 3 & 1 & 3 \\ -3 & -3 & -5 \end{bmatrix}$$
   (a) Determine the rational canonical form of A.
   (b) Determine the Jordan canonical form of A.

2. If
$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix},$$
   then prove that there does not exist a matrix with $N^2 = A$.

3. Let A be a real $n \times n$ matrix which is symmetric, i.e. $A^t = A$. Prove that A is diagonalizable.

4. Give an example of two nilpotent matrices A and B such that
   (a) A is not similar to B,
   (b) A and B have the same characteristic polynomial,
   (c) A and B have the same minimal polynomial, and
   (d) A and B have the same rank.

January 1994

1. Let A be an $n \times n$ matrix over a field F. Show that F^n is the direct sum of the null space and the range of A if and only if A and A^2 have the same rank.

2. Let A and B be $n \times n$ matrices over a field F.
   (a) Show AB and BA have the same eigenvalues.
   (b) Is AB similar to BA? (Justify your answer).

3. Given an exact sequence of finite-dimensional vector spaces
$$0 \xrightarrow{T_0} V_1 \xrightarrow{T_1} V_2 \xrightarrow{T_2} \cdots \xrightarrow{T_{n-2}} V_{n-1} \xrightarrow{T_{n-1}} V_n \xrightarrow{T_n} 0$$
   that is the range of T_i is equal to the null space of T_{i+1}, for all i. What is the value of $\sum_{i+1}^{n}(-1)^i \dim(V_i)$? (Justify your answer).

4. Let F be a field with q elements and V be a n-dimensional vector space over F.
   (a) Find the number of elements in V.
   (b) Find the number of bases in of V.
   (c) Find the number of invertible linear operators on V.

5. Let A and B be $n \times n$ matrices over a field F. Suppose that A and B have the same trace and the same minimal polynomial of degree $n-1$. Is A similar to B? (Justify your answer.)

6. Let $A = [a_{ij}]$ be an $n \times n$ matrix with $a_{ij} = 1$ for all i and j. Find its characteristic and minimal polynomial.

August 1994

1. Give an example of a matrix with real entries whose characteristic polynomial is $x^5 - x^4 + x^2 - 3x + 1$.

2. TRUE or FALSE. (If true prove it. If false give a counterexample.) Let A and B be $n \times n$ matrices with minimal polynomial x^4. If $\operatorname{rank} A = \operatorname{rank} B$, and $\operatorname{rank} A^2 = \operatorname{rank} B^2$, then A and B are similar.

3. Suppose that T is a linear operator on a finite-dimensional vector space V over a field F. Prove that the characteristic polynomial of T is equal to the minimal polynomial of T if and only if

$$\{U \in L(V,V) : TU = UT\} = \{f(T) : f \in F[x]\}.$$

January 1995

1. (a) Prove that if A and B are 3×3 matrices over a field F, a necessary and sufficient condition that A and B be similar over F is that that have the same characteristic and the same minimal polynomial.
   (b) Give an example to show this is not true for 4×4 matrices.

2. Let V be the vector space of $n \times n$ matrices over a field. Assume that f is a linear functional on V so that $f(AB) = f(BA)$ for all $A, B \in V$, and $f(I) = n$. Prove that f is the trace functional.

3. Suppose that N is a 4×4 nilpotent matrix over F with minimal polynomial x^2. What are the possible rational canonical forms for n?

4. Let A and B be $n \times n$ matrices over a field F. Prove that AB and BA have the same characteristic polynomial.

5. Suppose that V is an n-dimensional vector space over F, and T is a linear operator on V which has n distinct characteristic values. Prove that if S is a linear operator on V that commutes with T, then S is a polynomial in T.

August 1995

1. Let A and B be $n \times n$ matrices over a field F. Show that AB and BA have the same characteristic values in F.

2. Let P and Q be real $n \times n$ matrices so that $P + Q = I$ and $\operatorname{rank}(P) + \operatorname{rank}(Q) = n$. Prove that P and Q are projections. (Hint: Show that if $Px = Qy$ for some vectors x and y, then $Px = Qy = 0$.)

3. Suppose that A is an $n \times n$ real, invertible matrix. Show that A^{-1} can be expressed as a polynomial in A with real coefficients and with degree at most $n - 1$.

4. Let

$$A = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix}.$$

Determine the rational canonical form and the Jordan canonical form of A.

5. (a) Give an example of two 4×4 nilpotent matrices which have the same minimal polynomial but are not similar.

   (b) Explain why 4 is the smallest value that can be chosen for the example in part (a), i.e. if $n \le 3$, any two nilpotent matrices with the same minimal polynomial are similar.

Some Other Problems

1. **This is a very basic and important fact.** Let V be a finite dimensional vector space and f and g two linear functionals on V. If $\ker f = \ker g$ show g is a scalar multiple of f.

2. This problem makes explicit some facts that are used several times in solving some of the problems above.

   (a) Prove that if V is a finite dimensional vector space over the field F and $T \in L(V,V)$ and V is cyclic for T that any $S \in L(V,V)$ that commutes with T is a polynomial in T. That is $ST = TS$ implies that $S = p(T)$ for some $p(x) \in F[x]$. Hint: Let $\dim V = n$. Then because V is cyclic for T there is a vector $v_0 \in V$ so that $v_0, Tv_0, \dots, T^{n-1}v_0$ is a basis for V. Thus there are scalars $a_0, a_1, \dots, a_{n-1}$ so that $Sv_0 = a_0v_0 + a_1Tv_0 + a_2T^2v_0 + \cdots + a_{n-1}T^{n-1}v_0$. Then letting $p(x) = a_0 + a_1x + a_2x^2 + \cdots a_{n-1}x^{n-1}$ we have $Sv_0 = p(T)v_0$. Now use that S commutes with T (and thus also $p(T)$) to show that $ST^iv_0 = p(T)T^iv_0$ for $i = 0, 1, \dots, n-1$. Thus the two linear maps S and $p(T)$ agree on a basis and whence are equal.

   (b) If the minimal polynomial $f(x)$ of T has $\deg f(x) = \dim V$ then V is cyclic for T. Hint: I don't know any particularly easy way to do this. The basic idea is to factor $f(x) = p_1(x)^{k_1} \cdots p_l(x)^{k_l}$ into powers of primes and consider the corresponding primary decomposition $V = \ker(p_1(T)^{k_1}) \oplus \cdots \oplus \ker(p_l(T)^{k_l})$ and show that if $\deg f(x) = \dim V$ then each of the primary factors $\ker(p_i(T)^{k_i})$ is cyclic (this in turn uses that each of the $\ker(p_i(T)^{k_i})$ is a sum of cyclic subspaces). Now let v_i be a cyclic for T in $\ker(p_i(T)^{k_i})$ for $i = 1, \dots, l$. Then show the vector $v_0 = v_1 + v_2 + \cdots + v_l$ is cyclic for T.

3. Let $\lambda_1, \lambda_2, \dots, \lambda_n$ be distinct elements of the field F. Then the matrix

$$A = \begin{bmatrix} 1 & 1 & \cdots & 1 \\ \lambda_1 & \lambda_2 & \cdots & \lambda_n \\ \lambda_1^2 & \lambda_2^2 & \cdots & \lambda_n^2 \\ \vdots & \vdots & \ddots & \vdots \\ \lambda_1^{n-1} & \lambda_2^{n-1} & \cdots & \lambda_n^{n-1} \end{bmatrix}$$

is invertible. Hint: If A is singular then it has rank less than n and thus there is a nontrivial linear relation between the rows of A. This would in turn imply that there is a nonzero polynomial $p(x)$ of degree $\le n-1$ than had the n scalars $\lambda_1, \lambda_2, \dots, \lambda_n$ as roots. But this is impossible.

4. **This is anther set of facts that anyone who has had a graduate linear algebra class should know.** Let D be a diagonal metric that has all its diagonal elements distinct. That is

$$D = \operatorname{diag}(\lambda_1, \lambda_2, \dots, \lambda_n) := \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{bmatrix} \qquad \text{where } \lambda_i \neq \lambda_j \text{ for } i \neq j.$$

Then show

(a) The only matrices that compute with D are diagonal matrices.

(b) If A is any other diagonal matrix then C is a polynomial in D. That is there is a polynomial $p(x)$ so that $A = p(D)$.

(c) If A is any matrix that commutes with D then A is a polynomial in D.

(d) There is a cyclic vector for T. Hint: Let $e_1, \dots, e_n$ be the standard coordinate vectors. Then as D is diagonal $De_i = \lambda_i e_i$. Let $v = a_1e_1 + a_2e_2 + \cdots + a_ne_n$. Then show that v is a cyclic vector for D if and only if $a_i \neq 0$ for all i (One way to do this is use the last problem). In particular $v = e_1 + e_2 + \cdots + e_n$ is a cyclic vector for T.

5. This is anther standard problem. Let V be a finite dimensional vector space over a field F and let $T \in L(V, V)$. Let λ be an eigenvalue of T and let $V_\lambda := \{v \in V : Tv = \lambda v\}$ be the corresponding eigenspace.

(a) Let $S \in L(V, V)$ commute with T. Then show that V_λ is invariant under S. (That is show $v \in V_\lambda$ implies $Sv \in V_\lambda$.)

(b) Show that if A and B are $n \times n$ matrices over the complex numbers that commute they have a common eigenvector. Hint: As A is a complex matrix it has at least one eigenvalue λ. Let V_λ be the corresponding eigenspace. Then by what we have just done V_λ is invariant under B. But then the restriction of B to V_λ has an eigenvector in V_λ.

(c) This is a different way of looking at Problem 4 above. Assume V has an basis of eigenvectors $e_1, e_2, \dots, e_n$ of eigenvectors of T, that is $Te_i = \lambda e_i$. Also assume the eigenvalues are distinct: $\lambda_i \neq \lambda_j$ for $i \neq j$. Then show if S commutes with T then for some scalars c_i there holds $Se_i = c_ie_i$, and thus S is also diagonal in the basis $e_1, \dots, e_n$. Hint: Let $V_{\lambda_i} := \{v : Te_i = \lambda_i v\}$. Then by the assumptions V_{λ_i} is one dimensional with basis e_i. Part (a) of this problem then implies that V_{λ_i} is invariant under S. As V_{λ_i} is one dimensional this in turn implies e_i is an eigenvector of S.