SOME RESULTS ABOUT $C(K)$.

The goal of this set of exercises is to give show how some of the topics we have covered (the Stone-Weierstrass Theorem, weak topologies) can be combined to prove some results about $C(K)$, the algebra of continuous real valued functions on a compact Hausdorff space K.

Let K be a compact Hausdorff space and $C(K)$ the set of continuous functions $f\colon K \to \mathbf{R}$. We define norm on $C(K)$ by

$$\|f\|_{L^\infty} = \sup_{x\in K} |f(x)|.$$

Then $C(K)$ is a complete metric space with the distance between f and g being $\|f-g\|_{L^\infty}$. The space $C(K)$ is also an algebra with the usual pointwise product of functions. Note that

$$\|fg\|_{L^\infty} \le \|f\|_{L^\infty}\|g\|_{L^\infty}.$$

Recall from algebra that an ***ideal*** in $C(K)$ is a subset I such that $f_1, f_2 \in I$, then so is $f_1 + f_2 \in I$. And if $f \in I$ and $g \in C(K)$, then $fg \in I$. That is I is close under addition and by multiplication by elements of $C(K)$. Note that as $C(K)$ contains the constants we have that if $c_1, c_2 \in \mathbf{R}$ and $f_1, f_2 \in I$ then $c_1 f_1 + c_2 f_2 \in I$. Therefore I vector subspace of $C(K)$. It is therefore a subalgebra of $C(K)$. Also recall that I is a ***maximal ideal*** of $C(K)$ if it is an idea, $I \ne C(K)$, and if J is any ideal of $C(K)$ with $C(K)$ with $J \supseteq I$ and $J \ne I$, then $J = C(K)$.

Proposition 1. *Let $x_0 \in K$. Then $I_{x_0} := \{f \in C(K) : f(x_0) = 0\}$ is a maximal idea in $C(K)$.*

Problem 1. Prove this. □

These are all the only maximal ideals of $C(K)$:

Theorem 2. *If I is a maximal ideal in $C(K)$, then there is a unique point $x_0 \in K$ such that $I = I_{x_0}$.*

Problem 2. Prove this. HINT: If there is no point of K where all functions of I vanish, then use the form of the Stone-Weierstrass given in class to show that the closure of I is all of $C(K)$. That would mean that there is an $f \in K$ such that $\|f-1\|_{L^\infty} < 1/2$. But then f does not vanish and therefore $1/f \in C(K)$. This implies that $1 = f(1/f) \in I$ which in turn implies $I = C(K)$, which is impossible. Therefore there is at least one point x_0 such that $f(x_0) = 0$ for all $f \in I$. That is $I \subseteq I_{x_0}$. Now use the maximality of I. □

A ***multiplicative linear functional*** on $C(K)$ is a linear map $\alpha\colon C(K) \to \mathbf{R}$ such that $\alpha(fg) = \alpha(f)\alpha(g)$. As an example of such a function show that if $x_0 \in K$, then the evaluation map $\alpha(f) = f(x_0)$ is a nonzero linear multiplicative linear functional. We now show this is all of them.

Theorem 3. *Let $\alpha\colon C(K) \to \mathbf{R}$ be a nonzero linear multiplicative linear functional on $C(K)$. Then there is a unique $x_\alpha \in K$ such that α is given by $\alpha(f) = f(x_\alpha)$. That is all the nonzero linear multiplicative linear functionals are given by evaluations at points.*

Problem 3. Prove this. HINT: Show the kernel, $\ker(\alpha) := \{f : \alpha(f) = 0\}$, is a maximal idea in $C(K)$. Therefore by, Theorem 2, there is an x_0 so that $\ker(\alpha) = I_{x_0}$. Then show that $\alpha(f) = f(x_0)$. □

Recall that a map $F\colon C(K_2) \to C(K_1)$ is an ***algebra homomorphism*** iff F is linear and $F(fg) = F(f)F(g)$ for all $f, g \in C(K_2)$. Now let K_1 and K_2 be two compact Hausdorff spaces and $\varphi\colon K_1 \to K_2$ a continuous map. Define $\varphi^*\colon C(K_2) \to C(K_1)$ by

$$\varphi^*(f) = f \circ \varphi.$$

(Note the reverse of order: $\varphi\colon K_1 \to K_2$ but $\varphi^*\colon C(K_2) \to C(K_1)$.)

Proposition 4. *The map $\varphi^*\colon C(K_2) \to C(K_1)$ is an algebra homomorphism that satisfies $\varphi^* 1 = 1$ and $\|\varphi^* f\|_{L^\infty} \leq \|f\|_{L^\infty}$.*

Problem 4. Prove this. □

We now show that the converse of this is true. The fist step is:

Proposition 5. *Let K be a compact Hausdorff space and let $\mathcal{T}$ be the topology of K. Let $\mathcal{T}_{\mathrm{wk}}$ be the weak topology on K generated by the functions $C(K)$. Then $\mathcal{T} = \mathcal{T}_{\mathrm{wk}}$.*

Problem 5. Prove this. □

Proposition 6. *Let K_1 and K_2 be compact Hausdorff spaces and $\varphi\colon K_1 \to K_2$ a function. Then φ is continuous if and only if $f \circ \varphi \in C(K_1)$ for all $f \in C(K_2)$. (That is φ is continuous if and only if for all continuous functions $f \in K_2 \to \mathbf{R}$ the function $f \circ \varphi$ is a continuous function on K_1.)*

Problem 6. Prove this. HINT: The topology on K_2 is the weak topology generated by the functions in $C(K_2)$. Now use the second proposition on page 35 of the class notes. □

Theorem 7. *Let $F\colon C(K_2) \to C(K_1)$ be an algebra homomorphism. Then show there is a unique continuous map $\varphi\colon K_1 \to K_2$ such that $F(f) = \varphi^* f$.*

Problem 7. Prove this. HINT: For each $x \in K_1$ define a map $\beta_x\colon C(K_2) \to \mathbf{R}$ by $\beta_x(f) = (F(f))(x)$. Show this is a nonzero multiplicative linear functional. By Theorem 3 there is a unique $\varphi(x) \in K_2$ such that $\beta_x(f) = f(\varphi(x))$. This defines a function $\varphi\colon K_1 \to K_2$ and form the definition $(F(f))(x) = f(\varphi(x))$. That is $F(f) = f \circ \varphi = \varphi^* f$. It remains to show that φ is continuous (use Proposition 6) and unique. □