Applications of Tychonoff's Theorem.

This set of notes and problems is to show some applications of the Tychonoff product theorem. In the cases here we will have a set A be looking at $[a,b]^A$, that is the set of all functions form A to the interval $[a,b]$. This is the same as the product $\prod_{\alpha\in A} X_\alpha$ where $X_\alpha = [a,b]$ for all α. The product topology on $[a,b]^A$ has a base the sets of the form

$$U(a_1,t_1,\ldots,a_n,t_n;\varepsilon) := \{f \in [a,b]^A : |f(a_i) - t_1| < \varepsilon \text{ for } i = 1,\ldots,n\}$$

where $\{a_1,\ldots,a_n\}$ is a finite subset of A, $t_1,\ldots,t_n \in [a,b]$ and $\varepsilon > 0$. Tychonoff's theorem tells use that this topology on $[a,b]^A$ is compact and therefore if $\mathcal{F}$ is a closed subset of $[a,b]^A$ is a closed subset of $[a,b]^A$, then $\mathcal{F}$ is also compact. Here are couple of examples of this type.

Definition 1. A ***normed vector space*** is a vector space X over the field of real numbers $\mathbf{R}$ and along with a norm, which is a function $\|\cdot\|\colon X \to \mathbf{R}$ such that

1. $\|x\| \geq 0$ with $\|x\| = 0$ if and only if $x = 0$,
2. $\|x+y\| \leq \|x\| + \|y\|$ (i.e. the triangle inequality holds), and
3. for all $c \in \mathbf{R}$ and $x \in X$ we have $\|cx\| = |c|\|x\|$.

Definition 2. A ***bounded linear functional*** on the normed linear space X is a linear function $f\colon X \to \mathbf{R}$ such that for some $C \geq 0$

$$|f(x)| \leq C\|x\|$$

holds for all $x \in X$. Let X^* be the set of all bounded linear functionals $f\colon X \to \mathbf{R}$.

We now let

$$B := \{x \in X : \|x\| \leq 1\}.$$

Let

$$B^* := \{f\big|_B : f \in X^*, \text{ and for all } x \in X\ |f(x)| \leq \|x\|\}.$$

The inequality $|f(x)| \leq \|x\|$ implies that if $x \in B$, then $|f(x)| \leq \|x\| \leq 1$. Therefore B^* is a set of functions that map B into $[-1,1]$. That is B^* is a subset of $[-1,1]^B$.

Theorem 3. *For any normed linear space X, the set B^* is a closed subset of $[-1,1]^B$ and therefore B^* is compact with the topology it gets as a subspace of $[-1,1]^B$. (In functional analysis this topology is called the **weak* topology**.)*

Problem 1. Prove this.

Definition 4. A ***normed algebra*** is a normed linear space A with norm $\|\cdot\|$ such that A is has a product $(x,y) \mapsto xy$ that makes A into a associative algebra and such that $\|xy\| \leq \|x\|\|y\|$. A ***multiplicative linear functional*** on A is a linear function $f\colon A \to \mathbf{R}$ such that $f(xy) = f(x)f(y)$ and $|f(x)| \leq \|x\|$.

Let $B := \{x \in A : \|x\| \leq 1\}$ be the unit ball of A and let Δ be the set of the restrictions of multiplicative linear functionals to B. That is

$$\Delta := \{f\big|_B : f \text{ is a mulitiplictive linear functional}\}.$$

This is a subset of $[-1,1]^B$.

Theorem 5. *For any normed algebra the set Δ is a closed subset of $[-1,1]^B$ and therefore compact.*

Problem 2. Prove this.