Mathematics 242 Test #3, Take Home Portion

(1) (25 points) Solve

$$x'(t) + 5x(t) = \begin{cases} 0, & t < 2; \\ 12e^{3t}, & 2 \le t. \end{cases}$$

and $x(0) = 6$

Solution: The initial value problem can be rewritten as

$$x'(t) + 5x(t) = 12u(t - s)e^{3t}, \qquad x(0) = 6.$$

Let $X(s) = \mathcal{L}\{x(t)\}$. Then taking Laplace transforms gives and using $\mathcal{L}\{x'(t)\} = sX(s) - x(0) = sX(s) - 6$.

$$sX(s) - 6 + 5X(s) = \mathcal{L}\{12u(t-2)e^{3t}\} = \mathcal{L}\{12u(t-2)e^{3(t-2)+6}\} = 12e^6 \frac{e^{-2s}}{s-3}$$

where we have used the formula

$$\mathcal{L}\{u(t-a)f(t-a)\} = e^{-as}\mathcal{L}\{f(t)\}.$$

Solving for $X(s)$ gives

$$X(s) = \frac{6}{s+5} + e^6 e^{-2s}\left(\frac{12}{(s-3)(s+5)}\right) = \frac{6}{s+5} + \frac{3}{2}e^6 e^{-2s}\left(\frac{1}{s-3} - \frac{1}{s+5}\right)$$

Taking inverse Laplace transforms gives

$$\boxed{x(t) = 6e^{-5t} + \frac{3e^6}{2}u(t-2)\left(e^{3(t-2)} - e^{-5(t-2)}\right).}$$

(2) (20 points) Let $x(t)$ and $y(t)$ be related by

$$x'(t) = 2x(t) - 3y(t)$$
$$y'(t) = -3x(t) + 2y(t)$$

and

$$x(0) = 4, \quad y(0) = -6.$$

Take the Laplace transform of these equation to get a system of algebraic equations for $\mathcal{L}\{x\}$ and $\mathcal{L}\{y\}$. Solve these equations for $\mathcal{L}\{x\}$ and $\mathcal{L}\{y\}$ and then take the inverse Laplace transforms to find $x(t)$ and $y(t)$.

Solution: Let $X(s) = \mathcal{L}\{x(t)\}$ and $Y(s) = \mathcal{L}\{y(t)\}$. Using $\mathcal{L}\{x(t)\} = sX(s) - x(0) = sX(s) - 4$ and $\mathcal{L}\{y(t)\} = sY(s) - x(0) = sY(s) + 6$ we get

$$sX(s) - 4 = 2X(s) - 3Y(s)$$
$$sY(s) + 6 = -3X(s) + sY(s)$$

which can be rewritten as

$$(s-2)X(s) + 3Y(s) = 4$$
$$3X(s) + (s-2)Y(s) = -6$$

These can be solved by Crammer's rule to give

$$X(s) = \frac{4s+10}{(s+1)(s-5)} = \frac{5}{s-5} - \frac{1}{s+1},$$
$$Y(s) = \frac{-6s}{(s+1)(s-5)} = \frac{-5}{s-5} - \frac{1}{s+1}.$$

Taking inverse Laplace transforms then gives

$$\boxed{x(t) = 5e^{5t} - e^{-t}, \qquad y(t) = -5e^{5t} - e^{-t}.}$$

Here is a mistake that was make on the last test that you should not make again. The Laplace transform of a product is not the product of the Laplace transforms. That is

$$\mathcal{L}\{f(t)g(t)\} \neq \mathcal{L}\{f(t)\}\mathcal{L}\{g(t)\}.$$

Here is an example. Let $f(t) = e^{at}$ and $g(t) = e^{bt}$. Then

$$\mathcal{L}\{f(t)g(t)\} = \mathcal{L}\{e^{at}e^{bt}\} = \mathcal{L}\{e^{(a+b)t} = \frac{1}{s-(a+b)}$$

and

$$\mathcal{L}\{f(t)\}\mathcal{L}\{g(t)\} = \mathcal{L}\{e^{at}\}\mathcal{L}\{e^{bt}\} = \frac{1}{s-a}\frac{1}{s-b} = \frac{1}{(s-a)(s-b)}.$$

The two are clearly not equal.