Math 554, Test 2 Name: Key

Here is the information on the first test. 22 people took the exam. The high scores were 98 94, 94, 93, 90. The average was 73.05 with a standard deviation of 19.86. The median was 75. The break down in the grades is in the table.

Grade	Range	Number	Percent
A	80–100	10	44.45%
B	70–79	4	18.16%
C	60–69	4	18.18%
D	50–59	2	9.09%
F	0–59	2	9.09%

Problem 1. (a) State the ***intermediate value theorem***.

Solution: Let $f\colon [a,b] \to \mathbb{R}$ be continuous and c a number between $f(a)$ and $f(b)$. Then there is a $x_0 \in (a,b)$ with $f(x_0) = c$. $\square$

(b) Show that the function $f\colon \mathbb{R} \to \mathbb{R}$ given by

$$f(x) = 3x + 5\cos(x^6)$$

is onto.

Solution: The function f is continuous. To show that f is onto we need to show that for all $c \in \mathbb{R}$ there is an x_0 with $f(x_0) = c$. Let

$$a = \frac{c}{3} + 2, \qquad b = \frac{c}{3} - 2.$$

Then, as $-1 \le \cos(x) \le 1$

$$f(a) = 3\left(\frac{c}{3} - 2\right) + 5\cos(a^3)) = c - 6 + 5\cos(a^3) \le c - 6 + 5 = c - 1$$

and

$$f(b) = 3\left(\frac{c}{3} + 2\right) + 5\cos(b^3)) = c + 6 + 5\cos(b^3) \ge c + 6 - 5 = c + 1.$$

Therefore

$$f(a) \le c - 1 < c < c + 1 \le f(b)$$

and thus, by the intermediate value theorem, there is a $x_0 \in (a,b)$ with $f(x_0) = c$. $\square$

Alternative solution: Let n be a positive integer. Then

$$f(n) = 3n + 5\cos(n^3) \ge 3n - 5$$

and

$$f(-n) = -3n + 5\cos(n^3) \le -3n + 5.$$

For any $y_0 \in \mathbb{R}$ there will be an n such that $f(n) \le -3n + 5 < y_0 < 3n - 5 \le f(n)$ (just choose $n > \max\{-(y_0 - 5)/3, (y_0 + 5)/3\}$ by the Archimedian Principle) and f is continuous on the interval $[-n, n]$.

Thus by the intermediate value theorem there is a $x_0 \in (-n, n)$ with $f(x_0) = y_0$. As y_0 was any element of $\mathbb{R}$ this shows that f is onto. $\square$

Problem 2. Show that if $f\colon [a,b] \to [A,B]$ is continuous, strictly decreasing, onto then the inverse $f^{-1}\colon [A,B] \to [a,b]$ is also strictly decreasing.

Solution: Let $y_1, y_2 \in [A,B]$ with $y_1 < y_2$. We need to show that $f^{-1}(y_1) > f^{-1}(y_2)$. Assume toward a contradiction that this is false. Then $f^{-1}(y_1) \le f^{-1}(y_2)$. As f is decreasing this implies

$$y_1 = f(f^{-1}(y_1)) \ge f(f^{-1}(y_2)) = y_2$$

which contradicts $y_1 < y_2$. $\square$

Problem 3. (a) Define what it means for the function f to be ***uniformly continuous*** on an interval $[a,b]$.
Solution: The function f on $[a,b]$ is ***uniformly continuous*** iff for all $\varepsilon > 0$ there is a $\delta > 0$ such that for all $x, y \in [a,b]$

$$|x - y| < \delta \quad \text{implies} \quad |f(x) - f(y)| < \varepsilon.$$

$\square$

(b) State the theorem we covered about when some functions are uniformly continuous on closed bounded intervals.
Solution: A continuous function on a closed bounded interval $[a,b]$ is uniformly continuous on $[a,b]$. $\square$

(c) Use this theorem to explain briefly (at most a few sentences) why the function

$$f(x) = \sqrt{\frac{1+x^3}{x^4+9}}$$

is uniformly continuous on $[0,1]$.

Solution: The function f is continuous. In a little more detail if $g(x) = 1 + x^3$, $h(x) = x^4 + 9$ and $p(x) = \sqrt{x}$, then g and h are continuous on all of $\mathbb{R}$ as they are polynomials and thus g/h is continuous on $\mathbb{R}$ as h is never zero. The rational function g/h positive on $[0,1]$ and p is continuous on $[0,\infty)$. Therefore the composition

$$f(x) = p\left(\frac{g(x)}{h(x)}\right) = \sqrt{\frac{1+x^3}{x^4+9}}$$

is continuous. Now the theorem in part (b) lets us conclude that f is uniformly continuous. $\square$

Problem 4. (a) Define $\limsup_{x\to x_0} f(x) = \beta$.

Solution: Let

$$S_f(r; x_0) = \sup\{f(x) : 0 < |f(x) - f(x_0)| < r\}.$$

Then

$$\limsup_{x\to x_0} f(x) = \lim_{r\to x_0^+} S_f(r; x_0).$$

□

Remark 1. Some people gave the theorem used in the alternate solution to part (b) below as the definition. While strictly speaking this is not the definition of $\limsup_{x\to x_0} f(x) = \beta$, it still received full credit. □

(b) Assume that $\limsup_{x\to 0} f(x) = f(0) = 3$. Show 0 has a neighborhood N such that $f(x) < 4$ for all $x \in N$.

Solution: We are given that $\lim_{r\to 0^+} S_f(r; 0) = f(0) = 3$. Let $\varepsilon = 1$. Then there is a $\delta > 0$ so that

$$0 < r < \delta \implies |S_f(r; 0) - 3| < \varepsilon = 1.$$

In particular

$$0 < r < \delta \implies S_f(r; 0) < 3 + \varepsilon = 3 + 1 = 4.$$

Let $N = (-\delta, \delta)$ which is a neighborhood of 0. If $x \in (-\delta, \delta)$ there is an r such that $|x| < r < \delta$. Then

$$f(x) \le \sup\{f(x) : 0 < |x| < r\} = S_f(r; 0) < 4,$$

as required. □

Alternate Solution: We know the following:

Theorem. $\limsup_{x\to x_0} f(x) = \beta$ *if and only if for all* $\varepsilon > 0$ *there is a* δ *such that*

(a) $0 < |x - x_0| < \delta \implies f(x) < f(x_0) + \varepsilon$, *and*
(b) *there is an* x_1 *with* $0 < |x_1 - x_0| < \delta$ *and* $f(x_1) > f(x_0) - \varepsilon$.

Let $\varepsilon = 1$, then, as $\limsup_{x\to 0} f(x) = 3$, there is a $\delta > 0$ such that conditions (a) and (b) of the theorem hold with $x_0 = 0$. Let $N = (-\delta, \delta)$. Using part (a) we see that if $x \in N = (-\delta, \delta)$ then $f(x) < f(0) + \varepsilon = 3 + 1 = 4$. □

Problem 5. A function f is ***locally bounded above*** iff for all x in the domain of f there is a $\delta_x > 0$ and a constant M_x such that

$$f(y) \le M_x \qquad \text{for all} \qquad y \in (x - \delta_x, x + \delta_x).$$

Show that if f is locally bounded above on the closed bounded interval, $[a, b]$, then f is bounded above on $[a, b]$ (that is there is a constant M such that $f(x) \leq M$ on $[a, b]$).

Solution: For each $x \in [a, b]$ let $\delta_x > 0$ and M_x be so that

$$f(y) \leq M_x \qquad \text{for all} \qquad y \in (x - \delta_x, x + \delta_x).$$

Let

$$\mathcal{U} = \{(x - \delta_x, x + \delta_x) : x \in [a, b]\}.$$

Then each $(x - \delta_x, x + \delta_x)$ is open and this is a cover of $[a, b]$ as for any $x \in [a, b]$ we have $x \in (x - \delta_x, x + \delta_x) \in \mathcal{U}$. By the Heine-Borel Theorem there is a finite sub-cover $\{(x_1 - \delta_{x_1}, x + \delta_{x_1}), \dots, (x_n - \delta_{x_n}, x + \delta_{x_n})\} \subseteq \mathcal{U}$. Let

$$M = \max\{M_{x_1}, \dots, M_{x_n}\}.$$

If $x \in [a, b]$ then there is a j with $1 \leq j \leq n$ so that $x \in (x_j - \delta_{x_j}, x_j + \delta_{x_j})$ (this is as $\{(x_1 - \delta_{x_1}, x + \delta_{x.}), \dots, (x_n - \delta_{x_n}, x + \delta_{x_n})\}$ is a cover of $[a, b]$) and therefore $|x - x_j| < \delta_j$. Thus

$$f(x) \leq M_{x_j} \leq \max\{M_{x_1}, \dots, M_{x_n}\} = M.$$

As x was any element of $[a, b]$ this shows that $f(x)$ is bounded above on $[a, b]$. □

Problem 6. Show that if f is continuous on a neighborhood of $g(x_0)$ and g is continuous on a neighborhood of x_0 that

$$\lim_{x \to x_0} f(g(x)) = f(g(x_0)).$$

Solution: Let $\varepsilon > 0$. Then, as f is continuous, there is a $\delta_1 > 0$ such that

$$|y - g(x_0)| < \delta_1 \qquad \Longrightarrow \qquad |f(y) - g(x_0)| < \varepsilon. \tag{1}$$

As g is continuous there is a $\delta > 0$ such that

$$|x - x_0| < \delta \qquad \Longrightarrow \qquad |g(x) - g(x_0)| < \delta_1. \tag{2}$$

Then if $|x - x_0| < \delta$ we have $|g(x) - g(x_0)| < \delta_1$ by the implication (2). Therefore the implication (1) implies

$$|f(g(x)) - f(g(x_0))| < \varepsilon.$$

That is we have shown that for all $\varepsilon > 0$ there is a $\delta > 0$ such that $|f(g(x)) - f(g(x_0))| < \varepsilon$ when ever $|x - x_0| < \delta$. Therefore $\lim_{x \to x_0} f(g(x)) = f(g(x_0))$ □

Alternate solution: We know that if f and g are continuous and the composition $f \circ g$ is defined, then $f \circ g$ is continuous. Therefore

$$\lim_{x \to x_0} f(g(x)) = \lim_{x \to x_0} f \circ g(x) = f \circ g(x_0) = f(g(x_0)).$$

Actually this proof as it stands is not quit complete, as we need to show that $f \circ g$ is defined in some neighborhood of x_0. That is that the range of g is contained in the domain of f. To do this we note that f is defined on a neighborhood of $g(x_0)$. Let this neighborhood of $g(x_0)$ be $(g(x_0) - r, g(x_0) + r)$. As g is continuous for $\varepsilon = r$ there is a $\delta > 0$ such that

$$|x - x_0| < \delta \qquad \Longrightarrow \qquad |g(x) - g(x_0)| < r.$$

Thus if we restrict g to $(x_0 - \delta, x_0 + \delta)$ then $f \circ g$ is defined in the neighborhood $N = (x_0 - \delta, x_0 + \delta)$ of x_0 and now the argument above works.

As this was a timed exam I only took off 2 points if you gave this second solution and forgot to worry about if $f \circ g$ is defined. $\square$