Math 554 Homework

Here we will look at applications of

Theorem 1 (Intermediate Value Theorem)**.** *Let f be continuous on the closed interval $[a,b]$. Then for any β between $f(a)$ and $f(b)$ there is a $\beta \in (a,b)$ with $f(\beta) = c$.* $\square$

Problem 1*.* Show that any real number b, positive or negative, has at least one cube root. *Hint:* One way to start would be for show for any b the inequalities $-(|b|+1)^3 < b < (|b|+1)^3$ hold.

Problem 2 (Baby version of Brouwer Fixed Point Theorem)*.* Let $f\colon [a,b] \to [a,b]$ be continuous. (That is f is continuous on $[a,b]$ and $a \le f(x) \le b$. Show that there is a point $\beta \in [a,b]$ so that $f(\beta) = \beta$. (Such points are called ***fixed points***).

Problem 3*.* Let f continuous on all of $\mathbb{R}$ and assume that

$$|f(x)| \le \frac{|x| + 100}{2}$$

for all x. Show that f has a fixed point.

Problem 4*.* Show the equation

$$\cos(x) = \frac{1}{1+x^2}$$

has infinitely many solutions. You can assume that cos is continuous. *Hint:* To see what is going on, it will help to draw a picture.

Proposition 2. *Every cubic polynomial $p(x) = a_3x^3 + a_2x^2 + a_1x + a_0$ with $a_3 \ne 0$ has at least one root.*

Problem 5*.* Prove this along the following lines. We wish so solve

$$a_3x^3 + a_2x^2 + a_1x + a_0 = 0$$

by dividing by a_3 this is the same as solving

$$f(x) = \frac{p(x)}{a_3} = x^3 + b_2x^2 + b_1x + b_0 = 0$$

where $b_2 = a_2/a_3$, $b_1 = a_1/a_3$ and $b_0 = a_0/a_3$. Rewrite $f(x)$ as

$$f(x) = x^3\left(1 + \frac{b_2}{x} + \frac{b_1}{x^2} + \frac{b_0}{x^3}\right)$$

(a) If $|x| \ge 1$, show

$$\frac{1}{|x|^2} \le \frac{1}{|x|}, \qquad \frac{1}{|x|^3} \le \frac{1}{|x|}.$$

(b) If $|x| \ge 1$, show

$$\left|\frac{b_2}{x} + \frac{b_1}{x^2} + \frac{b_0}{x^3}\right| \le \frac{|b_2| + |b_1| + |b_0|}{|x|}.$$

(c) If $|x| \geq \max\{1, 2(|b_2| + |b_1| + |b_0|)\}$, show

$$\left|\frac{b_2}{x} + \frac{b_1}{x^2} + \frac{b_0}{x^3}\right| \leq \frac{1}{2}.$$

(d) f $|x| \geq \max\{1, 2(|b_2| + |b_1| + |b_0|)\}$, show

$$\frac{1}{2} \leq \left(1 + \frac{b_2}{x} + \frac{b_1}{x^2} + \frac{b_0}{x^3}\right) \leq \frac{3}{2}.$$

(e) Show

$$x \geq \max\{1, 2(|b_2| + |b_1| + |b_0|)\} \quad \Longrightarrow \quad f(x) > 0$$

and

$$x \leq -\max\{1, 2(|b_2| + |b_1| + |b_0|)\} \quad \Longrightarrow \quad f(x) < 0.$$

(f) Now show $f(x) = 0$ has at least one solution.

Problem 6. Note that the degree two polynomial $x^2 + 1$ has no real roots. We have just seen that all degree three polynomials have at least one real root. For which n is it true that all polynomials of degree n have a real root? You don't have to prove your answer, just explain why you think your answer is correct. Some pictures might help.