Math 554 Homework

Definition 1. A function f defined a set S is ***uniformly continuous*** on S iff for all $\varepsilon > 0$ there is a $\delta > 0$ so that for $x, y \in S$

$$|x - y| < \delta \qquad \Longrightarrow \qquad |f(x) - f(y)| < \varepsilon.$$

Problem 1. Show that the function

$$f(x) = \cos(1/x)$$

is not uniformly continuous on $(0,1)$. *Hint:* Towards a contradiction assume that f is uniformly continuous on $(0,1)$. Then for $\varepsilon = 1$ there is a $\delta > 0$ such that for all $x, y \in (0,1)$

$$|x - y| < \delta \qquad \Longrightarrow \qquad |f(x) - f(y)| < \varepsilon = 1.$$

Choose a positive integer n so large that $1/(2n\pi) < \delta$ (why can we do this?). Now let $x = 1/(2n\pi)$ and $y = 1/((2n+1)\pi)$ to get a contradiction.

Our big result about uniformly continuous functions (Theorem 2.2.12 in the text) is

Theorem 2. *If f is continuous on the closed bounded interval $[a,b]$ then f is uniformly continuous on $[a,b]$.*

Problem 2. Prove this along the following lines. This does not differ much from what we did in class today, but as this is the archetypal application of the Heine-Borel theorem it is good to review it. As f is continuous on $[a,b]$ for each $x \in [a,b]$ there is a $\delta_x > 0$ such that

$$|x - y| < \delta_x \qquad \Longrightarrow \qquad |f(x) - f(y)| < \frac{\varepsilon}{2}. \tag{1}$$

(a) Let
$$\mathcal{H} = \{(x - \delta_x/2, x + \delta_x/2) : x \in [a,b]\}$$
Explain why this is an open cover of $[a,b]$. (You can assume that $(x - \delta_x/2, x + \delta_x/2)$ is open.)

(b) Explain why there are finite number of points $x_1, x_2, \ldots, x_n \in [a,b]$ such that
$$[a,b] \subseteq \bigcup_{i=1}^{n} (x_i - \delta_{x_i}/2, x_i + \delta_{x_i}/2)$$
Hint: This should only take a sentence or two if you quote the right theorem.

Let
$$\delta = \min\left\{\delta_{x_1}/2, \delta_{x_2}/2, \ldots, \delta_{x_n}/2\right\}$$
and assume
$$x, y \in [a,b] \qquad \text{with} \qquad |x - y| < \delta.$$

(c) As $x \in [a,b] \subseteq \bigcup_{i=1}^{n}(x_i - \delta_{x_i}/2, x_i + \delta_{x_i}/2)$ there is a j with $x \in (x_j - \delta_{x_j}/2, x_j + \delta_{x_j}/2)$. Show

$$y \in (x_j - \delta_{x_j}, x_j + \delta_{x_j})$$

Hint: Remember $|x-y| < \delta$ and $\delta \leq \delta_j/2$.

(d) Show

$$|f(x) - f(x_j)| < \frac{\varepsilon}{2} \qquad \text{and} \qquad |f(y) - f(x_j)| < \frac{\varepsilon}{2}$$

Hint: The implication (1) is relevant.

(e) Conclude that

$$|f(x) - f(y)| < \varepsilon.$$

As x and y were any two points of $[a,b]$ with $|x-y| < \delta$ this finishes the proof.

The following problem indicates that this theorem has content even in special cases. In fact solving this problem is probably more work than proving the theorem. But looking at the proof of the theorem may give you an idea how to do it.

Problem 3. Show directly from the definition that the function f defined on $[0,1]$ by

$$f(x) = \sqrt{x}$$

is uniformly continuous. *Hint:* Don't get discouraged if you find this hard. It is not straightforward.

We now back track a bit. Read about ***monotone functions*** on pages 44–47 of the text. I recall one of the definitions here

Definition 3. The function $f\colon [a,b] \to \mathbb{R}$ is ***non-decreasing*** (also called ***monotone increasing***) iff

$$x_1 \leq x_2 \qquad \implies \qquad f(x_1) \leq f(x_2).$$

Theorem 4. *If f is non-decreasing on $[a,b]$ then for any $x_0 \in (a,b)$ both the one sided limits $\lim_{x \to x_0^-} f(x)$ and $\lim_{x \to x_0^+} f(x)$ exist.*

Problem 4. Prove that $\lim_{x \to x_0^-} f(x)$ exists. *Hint:* Let $S = \{f(x) : a \leq x < x_0\}$. Show that $S \neq \varnothing$ and S is bounded above (this should only be a sentence or two). Let $\beta = \sup S$. Now show $\lim_{x \to x_0^-} f(x) = \beta$.