Math 554 Homework

Definition 1. Let f be defined in a neighborhood of x_0. Then f is ***differentiable*** at x_0 iff the limit

$$f'(x_0) = \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0}$$

exists. The number $f'(x_0)$ is the ***derivative*** of f at x_0. □

Remark. This is equivalent to the existence of the limit

$$f'(x_0) = \lim_{h \to 0} \frac{f(x_0 + h) - f(x_0)}{h}$$

and you can use which ever of these you find more convenient. □

Theorem 2. *Let f be differentiable at x_0. Then f is continuous at x_0.*

Problem 1. Prove this along the following lines. First note that what we are given is that the limit

$$\lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0} = f'(x_0)$$

exists. Therefore, letting $\varepsilon = 1$, there is a $\delta_1 > 0$ such that

$$0 < |x - x_0| < \delta_1 \qquad \Longrightarrow \qquad \left| \frac{f(x) - f(x_0)}{x - x_0} - f'(x_0) \right| < \varepsilon = 1.$$

(a) Use the adding and subtracting trick to show

$$0 < |x - x_0| < \delta_1 \qquad \Longrightarrow \qquad \left| \frac{f(x) - f(x_0)}{x - x_0} \right| < |f'(x_0)| + 1.$$

(b) Now multiply by $|x - x_0|$ to get

$$0 < |x - x_0| < \delta_1 \qquad \Longrightarrow \qquad |f(x) - f(x_0)| < \big(|f'(x_0)| + 1\big)|x - x_0|.$$

(c) Let

$$\delta = \min\left\{\delta_1, \frac{\varepsilon}{|f'(x_0)| + 1}\right\}$$

and show

$$|x - x_0| < \delta \qquad \Longrightarrow \qquad |f(x) - f(x_0)| < \varepsilon$$

and thus f is continuous at x_0. □

We can now show that most of the rules we know and love from calculus hold. In class we showed the following

Theorem 3. *If f and g are defined in a neighborhood of x_0 and f and g are differentiable at x_0 then the product $p(x) = f(x)g(x)$ is differentiable at x_0 and the usual product rule*

$$p'(x_0) = f'(x_0)g(x_0) + f(x_0)g'(x_0)$$

holds.

Proof. We need to show that the limit $\lim_{x\to x_0} \frac{p(x)-p(x_0)}{x-x_0}$ exists. The proof is based on the adding and subtracting trick.

$$\begin{aligned}
&\lim_{x\to x_0} \frac{p(x)-p(x_0)}{x-x_0} = \lim_{x\to x_0} \frac{f(x)g(x)-f(x_0)g(x_0)}{x-x_0} \\
&= \lim_{x\to x_0} \frac{f(x)g(x)-f(x_0)g(x)+f(x_0)g(x)-f(x_0)g(x_0)}{x-x_0} \\
&= \lim_{x\to x_0} \left(\frac{f(x)-f(x_0)}{x-x_0} g(x) + f(x_0) \frac{g(x)-g(x_0)}{x-x_0} \right) \\
&= \lim_{x\to x_0} \frac{f(x)-f(x_0)}{x-x_0} \lim_{x\to x_0} g(x) + f(x_0) \lim_{x\to x_0} \frac{g(x)-g(x_0)}{x-x_0} \\
&= f'(x_0)g(x_0) + f(x_0)g'(x_0)
\end{aligned}$$

were at the last step we have used the continuity of g so that $\lim_{x\to x_0} g(x) = g(x_0)$. □

Theorem 4. *If $f_1, \ldots, f_n$ are all defined on a neighborhood of x_0 and each f_j is differentiable at x_0 then so is the product*

$$p = f_1 f_2 \cdots f_n = \prod_{k=1}^{n} f_k$$

and the derivative is given by

$$p'(x_0) = \sum_{k=1}^{n} f_1(x_0) \cdots f_{k-1}(x_0) f_k'(x_0) f_{k+1}(x_0) \cdots f_n(x_0).$$

(This sum has n terms and in the k-th term the derivative f_k' appears.)

Remark. For small values of n this is

$$\begin{aligned}
(f_1f_2f_3)' &= f_1'f_2f_3 + f_1f_2'f_3 + f_1f_2f_3' \\
(f_1f_2f_3f_4)' &= f_1'f_2f_3f_4 + f_1f_2'f_3f_4 + f_1f_2f_3'f_4 + f_1f_2'f_3f_4 + f_1f_2f_3f_4'.
\end{aligned}$$

If none of the functions $f_1, \ldots, f_n$ vanish at x_0 then the formula for $(f_1f_2\cdots f_n)'$ can be written as

$$(f_1f_2\cdots f_n)' = (f_1f_2\cdots f_n)\left(\frac{f_1'}{f_1} + \frac{f_2'}{f_2} + \cdots + \frac{f_n'}{f_n}\right)$$

or

$$\frac{(f_1f_2\cdots f_n)'}{f_1f_2\cdots f_n} = \frac{f_1'}{f_1} + \frac{f_2'}{f_2} + \cdots + \frac{f_n'}{f_n}$$

where all these expersion are evaluated at x_0. □

Problem 2. Prove Theorem 4. *Hint:* You should only have to use Theorem 3 and induction. □

Theorem 5. *If g is defined in a neighborhood of x_0, is differentiable at x_0, and $g(x_0) \neq 0$, then*

$$h = \frac{1}{g}$$

is differentiable at x_0 and

$$h'(x_0) = \frac{-g'(x_0)}{g(x_0)^2}.$$

Problem 3. Prove this. *Hint:* Note

$$\begin{aligned}
\frac{h(x) - h(x_0)}{x - x_0} &= \frac{1}{x - x_0}\left(\frac{1}{g(x)} - \frac{1}{g(x_0)}\right) \\
&= \frac{1}{x - x_0}\left(\frac{g(x_0) - g(x)}{g(x)g(x_0)}\right) \\
&= \frac{-\left(\dfrac{g(x) - g(x_0)}{x - x_0}\right)}{g(x)g(x_0)}
\end{aligned}$$

□

Theorem 6. *Let f and g be defined in a neighborhood of x_0. Assume that both f and g are differentiable at x_0 and that $g(x_0) \neq 0$. Then the quotient*

$$q(x) = \frac{f(x)}{g(x)}$$

is defined on a neighborhood of x_0 and is differentiable at x_0 with

$$q'(x_0) = \frac{f'(x_0)g(x_0) - f(x_0)g'(x_0)}{g(x_0)^2}.$$

Problem 4. Prove this. *Hint:* You should not even have to take a limit. Just write

$$q = f\left(\frac{1}{g}\right)$$

and use Theorems 3 and 5.

Problem 5. Show that the function

$$f(x) = \begin{cases} x^2\cos(1/x), & x \neq 0; \\ 0, & x = 0. \end{cases}$$

is differentiable at $x_0 = 0$. What is $f'(0)$? *Hint:* You night want to use the pinching theorem for limits to do this.