Math 554 Homework

Now that we have defined derivatives, the material willstart to look more familiar. We start with what is often called the "first dirivative test" for a local maximum or minimum.

Definition 1. Let f be defined in a neighborhood of x_0.

(a) f has a ***local maximum*** at x_0 iff there is a $r > 0$ such that $f(x) \leq f(x_0)$ for all $x \in (x_0 - r, x_0 + r)$.
(b) f has a ***local minimum*** at x_0 iff there is a $r > 0$ such that $f(x) \geq f(x_0)$ for all $x \in (x_0 - r, x_0 + r)$.
(c) f has a ***local extrema*** at x_0 iff f has either a local maximum or minimum at x_0.

Proposition 2. *If f has a local maximum at x_0, then $-f$ has a local minimum at x_0. If f has a local minimum at x_0, then $-f$ has a local maximum at x_0.*

Proof. Hopefully this is clear. □

Theorem 3. *Let f be defined on a neighborhood of x_0 and assume*

(a) *f is differentiable at x_0,*
(b) *f has a local extrema at x_0.*

then $f'(x_0) = 0$.

Problem 1. Prove this along the following lines. First note that we can assume that f has a local maximum at x_0, otherwise f has a local minimum at x_0, it which case replace f by $-f$. Let $r > 0$ be so that $f(x) \leq f(x_0)$ for $x \in (x_0 - r, x_0 + r)$. We are assuming that $f'(x_0)$ exists and therefore

$$f'(x_0) = \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0}.$$

Thus we also have the existence of the one sided limits

$$f'(x_0) = \lim_{x \to x_0^+} \frac{f(x) - f(x_0)}{x - x_0} = \lim_{x \to x_0^-} \frac{f(x) - f(x_0)}{x - x_0}$$

(a) Show if $x_0 < x < x_0 + r$ then

$$\frac{f(x) - f(x_0)}{x - x_0} \leq 0.$$

(b) Use (a) to show

$$f'(x_0) = \lim_{x \to x_0^+} \frac{f(x) - f(x_0)}{x - x_0} \leq 0$$

(c) Show if $x_0 - r < x < x_0$ then

$$\frac{f(x) - f(x_0)}{x - x_0} \geq 0.$$

(d) Use (c) to show

$$f'(x_0) = \lim_{x \to x_0^-} \frac{f(x) - f(x_0)}{x - x_0} \geq 0$$

(e) Show $f'(x_0) = 0$. □

Theorem 4 (Rolle's Theorem)**.** *Let f be continuous on $[a, b]$ and differentiable on (a, b) and assume $f(a) = f(b)$. Then there is a $\xi \in (a, b)$ with $f'(\xi) = 0$.*

Problem 2*.* Prove this along the following lines.

(a) Explain why it is enough to show that f has a local extrema in the open interval (a, b).
(b) Explain how we know that f achieves both its maximum and minimum on $[a, b]$. *Hint:* You can just quote a theorem.
(c) Show that if there is a some $x \in (a, b)$ with $f(x) > f(a)$ then f achieves its maximum at a point $\xi \in (a, b)$ and thus $f'(\xi) = 0$.
(d) Show that if there is some $x \in (a, b)$ with $f(x) < f(a)$ then f achieves its minimum at a point $\xi \in (a, b)$ and thus $f'(\xi) = 0$.
(e) Show that if neither of the cases (c) or (d) hold, then f is constant on $[a, b]$ and thus $f'(\xi) = 0$ for all $\xi \in (a, b)$. □

Theorem 5 (Mean Value Theorem)**.** *Let f be continuous on $[a, b]$ and differentiable on (a, b). Then there is a $\xi \in (a, b)$ such that*

$$f(b) - f(a) = f'(\xi)(b - a).$$

Problem 3*.* Prove this along the following lines. Let g be defined on $[a, b]$ by

$$g(x) = f(x) - \frac{f(b) - f(a)}{b - a}(x - a).$$

(a) Explain briefly why g satisfies the hypothesis of Rolle's theorem.
(b) By Rolle's theorem there is a $\xi \in (a, b)$ with $g'(\xi) = 0$. Show that $g'(\xi) = 0$ can be rearranged to give $f(b) - f(a) = f'(\xi)(b - a)$. □

The following is just a restatement of the Mean Value Theorem is a form that is sometimes a bit easier to apply to concrete cases.

Theorem 6 (Second form of the Mean Value Theorem)**.** *Let f be continuous on an interval I (we don't care if the interval is open, closed, half open, bounded or unbounded) and that f is differentiable on the interior of I. Let $x_1, x_2 \in I$ with $x_1 \neq x_2$. Then there is a point, ξ, between x_1 and x_2 such that*

$$f(x_2) - f(x_1) = f'(\xi)(x_2 - x_1).$$

□

To see this holds we can assume that $x_1 < x_2$. Then f is continuous on $[x_1, x_2]$ and differentiable on (x_1, x_2) and thus there is a $\xi \in (x_1, x_2)$ with $f(x_2) - f(x_1) = f'(\xi)(x_2 - x_1)$ by our first form of the mean value theorem. We now apply this to verify some standard facts from calculus.

Theorem 7. *Assume that f is continuous on an interval, I, and differentiable on the interior of I. Assume that $f'(\xi) = 0$ for all x in the interior of I. Then f is constant.*

Problem 4. Prove this using the mean value theorem. □

Theorem 8. *Assume that f is continuous on an interval, I, and differentiable on the interior of I.*

(a) *If $f' > 0$ on the interior of I then f is strictly in increasing on I.*
(b) *If $f' < 0$ on the interior of I then f is decreasing on I.*
(c) *If $f' \geq 0$ on the interior of I then f is monotone increasing on I.*
(d) *If $f' \leq 0$ on the interior of I then f is monotone decreasing on I.*

Problem 5. Prove part (a) from the Mean Value Theorem. (The proofs of the other parts are just about identical to this.) □

From now on, every time you see and expression $f(x_2) - f(x_1)$ (or $g(b) - g(a)$, or $h(y) - h(x)$) you should consider using the mean value theorem. For example suppose that you are ask to show

$$|\cos(2x) - \cos(2y)| \leq 2|x - y|$$

This contains an expression of the form $f(x) - f(y)$ where $f(x) = \cos(2x)$. So the mean value theorem comes to mind. In this case f is differentiable on all $\mathbb{R}$ and $f'(\theta) = -2\sin(2\theta)$. Thus

$$\begin{aligned} |\cos(2x) - \cos(2y)| &= |f(x) - f(y)| \\ &= |f'(\xi)||x - y| \\ &= |-2\sin(2\xi)||x - y| \\ &\leq 2|x - y| \end{aligned}$$

where ξ is between x and y and we have used that $|-2\sin(2\xi)| \leq 2$. (Yes, I know that we have yet to show that cos and sin are differentiable but we assume it here to have interesting examples.)

Problem 6. Show that for real numbers a, b

$$\left|\sqrt{b^2+1} - \sqrt{a^2+1}\right| \leq |b - a|.$$

(You can assume that square root function is differentiable with its is usual derivative.) □

Problem 7. Let f be defined on $\mathbb{R}$ by $f(x) = 5x + \cos(2x)$. Show for $x, y \in \mathbb{R}$ that

$$|f(x) - f(y)| \geq 3|x - y|.$$

□

Problem 8. Let f differentiable on $\mathbb{R}$ with $|f'(x)| \leq M$ for some constant M. Show $|f(x_2) - f(x_1)| \leq M|x_2 - x_1|$. □

There is a generalization of the mean value theorem that is sometimes useful.

Theorem 9 (Generalized Mean Value Theorem)**.** *Let f and g be continuous on $[a, b]$ and differentiable on (a, b). There there is a $\xi \in (a, b)$ such that*

$$(f(b) - f(a))g'(\xi) = (g(b) - g(a))f'(\xi).$$

(Note if $g(x) = x$, then this reduces to the usual mean value theorem.)

Problem 9*.* Prove this. *Hint:* Show the function

$$h(x) = (f(b) - f(a))g(x) - (g(b) - g(a))f(x)$$

satisfies the hypothesis of Rolle's theorem. □