Math 554 Homework

We now wish to look at one of the other standard topics in differential calculus, l'hôpital's rule. Recall this involves evaluating limits of the type

$$\lim_{x\to x_0} \frac{f(x)}{g(x)}$$

where $f(x_0) = g(x_0) = 0$ which leads to

$$\lim_{x\to x_0} \frac{f(x)}{g(x)} = \frac{0}{0}$$

which, at least formally, does not exist. Here is the basic result.

Theorem 1 (L'hôpital's rule)**.** *Let f and g be differentiable in a neighborhood of x_0. Assume that $f(x_0) = g(x_0) = 0$ and that*

$$\lim_{x\to x_0} \frac{f'(x)}{g'(x)} = L$$

exists. Then $\lim_{x\to x_0} f(x)/g(x)$ *exists and*

$$\lim_{x\to x_0} \frac{f(x)}{g(x)} = L$$

This is usually stated informally as that if $f(x_0) = g(x_0) = 0$ then

$$\lim_{x\to x_0} \frac{f(x)}{g(x)} = \lim_{x\to x_0} \frac{f'(x)}{g'(x)}.$$

The important part is that the existence of the limit $\lim_{x\to x_0} \frac{f'(x)}{g'(x)}$ implies the existence of the limit $\lim_{x\to x_0} \frac{f(x)}{g(x)}$.

The proof is based on

Theorem 2 (Generalized Mean Value Theorem)**.** *Let f and g be continuous on $[a, b]$ and differentiable on (a, b). There there is a $\xi \in (a, b)$ such that*

$$(f(b) - f(a))g'(\xi) = (g(b) - g(a))f'(\xi).$$

(Note if $g(b) - g(a) \neq 0$ and $g'(\xi) \neq 0$ this and be written as

$$\frac{f(b) - f(a)}{g(b) - g(a)} = \frac{f'(\xi)}{g'(\xi)}$$

which will be useful below.) □

which we proved in the last homework.

Problem 1. Prove Theorem 1 as follows. As $\varepsilon > 0$ then as $\lim_{x\to x_0} \frac{f'(x)}{g'(x)} = L$ there is a $\delta > 0$ such that

$$0 < |x - x_0| < \delta \quad \Longrightarrow \quad \left| \frac{f'(x)}{g'(x)} - L \right| < \varepsilon.$$

Let x be so that $0 < |x - x_0| < \delta$. Then, by the generalized mean value theorem, there is a ξ between x and x_0 such that

$$\frac{f(x)}{g(x)} = \frac{f(x) - 0}{g(x) - 0} = \frac{f(x) - f(x_0)}{g(x) - g(x_0)} = \frac{f'(\xi)}{g'(\xi)}.$$

Use this to show

$$0 < |x - x_0| < \delta \qquad \Longrightarrow \qquad \left| \frac{f(x)}{g(x)} - L \right| < \varepsilon$$

and thus $\lim_{x \to x_0} \frac{f(x)}{g(x)} = L$. (A main point is that $0 < |\xi - x_0| < \delta$, so be sure to explain why this holds.) $\square$

Here is a standard application of l'hôpital's rule:

$$\lim_{x \to 0} \frac{\sin(2x)}{3x} = \lim_{x \to 0} \frac{\sin(2x)'}{(3x)'} = \lim_{x \to 0} \frac{2\cos(2x)}{3} = \frac{2\cos(0)}{3} = \frac{2}{3}.$$

It can also be applied several times in a row:

$$\begin{aligned}
\lim_{x \to 0} \frac{1 - \cos(x)}{x^2} &= \lim_{x \to 0} \frac{\sin(x)}{2x} && \left(\text{take } \frac{d}{dx} \text{ of top and bottom}\right) \\
&= \lim_{x \to 0} \frac{\cos(x)}{2} && \left(\text{take } \frac{d}{dx} \text{ of top and bottom}\right) \\
&= \frac{\cos(0)}{2} \\
&= \frac{1}{2}.
\end{aligned}$$

So we have shown $\lim_{x \to 0} \frac{1-\cos(x)}{x^2} = \frac{1}{2}$. Note that in terms of showing this limit exists, this should be read from the bottom up. That is l'hôpital's rule shows that $\lim_{x \to 0} \frac{\sin(x)}{2x}$ exists as $\lim_{x \to 0} = \frac{\sin(x)'}{(2x)'} = \lim_{x \to 0} \frac{\cos(0)}{2} = \frac{1}{2}$ exists. Then anther application of l'hôpital's rule shows that $\lim_{x \to 0} \frac{1-\cos(x)}{x^2} = \lim_{x \to 0} \frac{(1-\cos(x))'}{(x^2)'} = \lim_{x \to 0} \frac{\sin(x)}{2x}$ exists.

Problem 2. For some practice using l'hôpital's use do problems 2, 3, 5, 6 on Page 96 of the text. Note that Trench has much more information about general forms of l'hôpital's rule. It is worth looking through the various forms of the result that he gives. $\square$

Now back to Rolle's theorem. First a definition.

Definition 3. Let f be defined on an open integral I. Then f is ***twice differentiable*** on I if f' exsits at all points of I and the function f' is differentiable on I. We denote the derivative of f' as f'' or $f^{(2)}$ and it is called the ***second derivative*** of f. If f'' exists at all points of I and f'' is differentiable on I its derivative is denoted by f''' or $f^{(3)}$ and is called the ***third derivative*** of f and f is said to be ***three times differentiable***. Continuing recursively, if we have defined what it means for f to be n ***times differentiable*** on I and the n***-th derivative***, $f^{(n)}$, is differentiable on I

then the derivative of $f^{(n)}$ is denoted by $f^{(n+1)}$ and f is $(n+1)$ ***times differentiable*** on I. $\square$

Remark. For consistency sake we set $f^{(0)} = f$ and $f^{(1)} = f'$ $\square$

Problem 3. Show that the function f on $\mathbb{R}$ defined by

$$f(x) = \begin{cases} x^2, & x \geq 0; \\ -x^2, & x < 0. \end{cases}$$

is differentiable on $\mathbb{R}$ but not twice differentiable. *Hint:* Show $f'(x) = 2|x|$. You may have to use the limit definition to compute $f'(0)$. $\square$

Problem 4. Find a function that is twice differentiable on $\mathbb{R}$ but not three times differentiable. $\square$

Proposition 4. *Let I be an open interval and assume f is twice differentiable on I. Let $x_0, x_1 \in I$ with $x_0 \neq x_1$. Assume $f(x_0) = f'(x_0) = 0$ and $f(x_1) = 0$. Then there is a point ξ between x_0 and x_1 with $f''(\xi) = 0$.*

Proof. As $f(x_0) = f(x_1) = 0$ by Rolle's Theorem there is a ξ_1 between x_0 and x_1 with $f'(\xi_1) = 0$. But the function f' is differentiable on I and $f'(x_0) = f'(\xi_1) = 0$ and thus anther application of Rolle's Theorem gives us a ξ between x_0 and ξ_1 with $f''(\xi) = (f')'(\xi) = 0$. As ξ_1 is between x_0 and x_1 and ξ is between x_0 and ξ_1 we have that ξ is between x_0 and x_1. $\square$

This generalizes

Theorem 5. *Let f be $n+1$ times differentiable on the open interval I. Let $x_0, x_1 \in I$ with $x_0 \neq x_1$. Assume that*

- $f(x_0) = f'(x_0) = \cdots = f^{(n)}(x_0) = 0,$
- $f(x_1) = 0.$

Then there is a point ξ between x_0 and x_1 with

$$f^{(n+1)}(\xi) = 0.$$

$\square$

Problem 5. Prove this. *Hint:* There are several ways to do this. One is to look at the proof of Proposition 4 and meditate upon induction. $\square$

Proposition 6. *Let f be twice differentiable on the open interval I and let $a, b \in I$ with $a \neq b$. Then there is a ξ between a and b with*

$$f(b) = f(a) + f'(a)(b-a) + \frac{f''(\xi)}{2}(b-a)^2.$$

Proof. Let h be defined on I by

$$h(x) = f(x) - f(a) - f'(a)(x-a) - c(x-a)^2$$

where c is a constant to be chosen shortly. Note

$$h(a) = 0$$

and

$$h'(x) = f'(x) - f'(a) - 2c(x-a),$$

and thus
$$h'(a) = 0.$$
With applying Proposition 4 in mind, we wish to choose c so that $h(b) = 0$. That is
$$h(b) = f(b) - f(a) - f'(a)(b-a) - c(b-a)^2 = 0$$
which leads to
$$c = \frac{f(b) - f(a) - f'(a)(b-a)}{(b-a)^2}.$$
With this choice of c we have $h(a) = h'(a) = h(b) = 0$ and thus by Proposition 6 there is a ξ between a and b with
$$h''(\xi) = 0.$$
By direct calculation
$$h''(x) = f''(x) - 2c.$$
Then $h''(\xi) = 0$ yields
$$f''(\xi) - 2c = 0.$$
But using the formula for c above we find
$$f''(\xi) - 2\left(\frac{f(b) - f(a) - f'(a)(b-a)}{(b-a)^2}\right) = 0$$
which can be rearranged to give
$$f(b) = f(a) + f'(a)(b-a) + \frac{f''(\xi)}{2}(b-a)^2$$
as required. □

As this was a more or less direct consequence of Proposition 4 it makes sense to look for a generalization that depends on Theorem 5. This requires a few preliminaries.

Lemma 7. *Let n be a non-negative integer and let p be defined on $\mathbb{R}$ by*
$$p(x) = \frac{(x-a)^k}{k!}.$$
Then the derivatives of p evaluated at a are given by
$$p^{(m)}(a) = \begin{cases} 1, & m = k; \\ 0, & m \neq k. \end{cases}$$

Problem 6. Prove this. □

Definition 8. Let f be n times differentiable on a neighborhood of a. Then the ***degree*** n ***Taylor polynomial*** of f at x is
$$T_n(x) := \sum_{k=0}^{n} f^{(k)}(a)\frac{(x-a)^k}{k!}.$$
□

Problem 7. Show that if f is n times differentiable on an open interval I and T_n is its degree n Taylor polynomial at a, then for $0 \le k \le n$

$$T_n^{(k)}(a) = f^{(k)}(a).$$

That is the k-th derivatives of T_n and f agree at a for $0 \le k \le n$. $\square$

Theorem 9 (Taylor's Theorem with Lagrange's form of the remainder)**.** *Let f be $(n+1)$ times differentiable on the open interval I and let $a, b \in I$ with $a \neq b$. Let T_n be the degree n Taylor polynomial of f at a. Then there is a ξ between a and b such that*

$$f(b) = T_n(b) + f^{(n+1)}(\xi)\frac{(b-a)^{n+1}}{(n+1)!}.$$

(The term $E_n(b) = f^{(n+1)}(\xi)\frac{(b-a)^{n+1}}{(n+1)!} = f(b) - T_n(b)$ is the ***error term*** *or* ***remainder term*** *when approximating f by its Taylor polynomial T_n.)*

Problem 8. Prove this along the lines of the proof of Proposition 6 as follows. Let h be defined on I by

$$h(x) = f(x) - T_n(x) - c(x-a)^{n+1}$$

where c is to be chosen so that $h(b) = 0$

(a) Show

$$h(a) = h'(a) = \cdots = h^{(n)}(a) = 0.$$

(b) Show $h(b) = 0$ if we choose c to be

$$c = \frac{f(b) - T_n(b)}{(b-a)^{n+1}}.$$

(c) Show there is a ξ between a and b such that

$$h^{(n+1)}(\xi) = 0.$$

Hint: Just note Theorem 5 applies.

(d) Show $h^{(n+1)}(\xi) = 0$ is the same as

$$0 = f^{(n+1)}(\xi) - c(n+1)!$$

and use this and the formula for c given in part (b) to complete the proof. $\square$

We restate this with slightly different notation (just replacing a and b with x_0 and x.)

Theorem 10 (Taylor's Theorem with Lagrange's form of the remainder, form 2)**.** *Let f be $(n+1)$ times differentiable on the open interval I and let $x, x_0 \in I$ with $x \neq x_0$. There there is a ξ betweem x and x_0 such that*

$$f(x) = \sum_{k=0}^{n} f^{(k)}(x_0)\frac{(x-x_0)^k}{k!} + f^{(n+1)}(\xi)\frac{(x-x_0)^{n+1}}{(n+1)!}. \qquad \square$$

Remark. In the case that $n = 0$ this becomes

$$f(x) = f(x_0) + f'(\xi)(x - x_0),$$

which can be rewritten as $f(x) - f(x_0) = f'(\xi)(x - x_0)$. That is for $n = 0$ we just get the mean value theorem. $\square$

One last restatement of Taylor's theorem. If we let $x = x_0 + h$ we get

$$f(x_0 + h) = \sum_{k=0}^{n} f^{(k)}(x_0)\frac{h^k}{k!} + f^{(n+1)}(\xi)\frac{h^{n+1}}{(n+1)!}$$

where ξ is between x_0 and $x_0 + h$.

As an examples of Taylor's theorem we have

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{e^{\xi}x^4}{4!} \qquad \text{(Used } n = 3.\text{)}$$

$$\cos(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{\cos(\xi)x^6}{6!} \qquad \text{(Used } n = 5.\text{)}$$

$$\sin(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{\cos(\xi)x^7}{7!} \qquad \text{(Used } n = 6.\text{)}$$

where ξ is between x and 0 (and of course the value of ξ is different in each of the three equations).