Math 554 Homework

You should know the statements of the **Heine-Borel Theorem** (Page 25 of the text) and the ***Bolzano-Weierstrass Theorem*** (Page 27) and the definition on an open cover.

Theorem 1 (Heine-Borel Theorem for intervals)**.** *Let H be an open cover of the closed bounded interval $[a, b]$. Then there are a finite number of open sets $U_1, \ldots, U_n \in H$ with $[a, b] \subseteq U_1 \cup \cdots \cup U_n$.*

Here is the a proof broken up into hopefully bite size steps. It $t \in [a, b]$, say that $[a, t]$ is finitely covered by H iff there are a finite number of open sets $U_1, \ldots, U_m \in H$ that cover $[a, t]$. (That is $[a, t] \subset U_1 \cup \cdots \cup U_m$.)

Problem 1. Let U be an open set and $t \in U$. Show that there is an $\varepsilon > 0$ so that $[t - \varepsilon, t + \varepsilon] \subseteq U$. *Hint:* As U is open we know that this is an open neighborhood $(t - \varepsilon_1, t + \varepsilon_1) \subset U$ for some $\varepsilon_1 > 0$. Let $0 < \varepsilon < \varepsilon_1$.

Problem 2. Show that if $t < b$ and $[a, t]$ is finitely covered by H then there is an $\varepsilon > 0$ so that $[a, t + \varepsilon]$ is also finite covered by H. *Hint:* As $[a, t]$ is finitely covered by H, we have $[a, t] \subseteq U_1 \cup \cdots \cup U_m$ for some $U_1, \ldots, U_m \in H$. The points t will be in one of the sets $U_1, \ldots, U_m$, say $t \in U_j$. As U_j is open by Problem 1 there is an $\varepsilon > 0$ such that $[t - \varepsilon, t + \varepsilon] \subseteq U_m$. Now explain why $[a, t + \varepsilon]$ is finitely covered by H.

Let

$$F = \{t : t \in [a, b] \text{ and } [a, t] \text{ is finitely covered by } H\}$$

and let

$$\beta = \sup F.$$

Problem 3. Show that $\beta \in F$. *Hint:* As H is a cover of $[a, b]$ and $\beta \in [a, b]$ there is a $U \in H$ with $\beta \in U$. As U is open there is an $\varepsilon > 0$ so that $[\beta - \varepsilon, \beta + \varepsilon] \subseteq U$. As $\beta = \sup F$, there is a $t \in F$ with $t \in [\beta - \varepsilon, \beta]$. Thus $[a, t]$ is covered by a finite number of sets $U_1, \ldots, U_m \in H$. Show that $U_1, \ldots, U_m, U$ covers $[a, \beta + \varepsilon]$ and thus $\beta \in F$.

Problem 4. Show that $\beta = b$ and that this implies the Theorem. *Hint:* Explain why this follows from the hint to the last problem.

Theorem 2 (Heine-Borel Theorem general case)**.** *Let H be an open cover a closed subset S of $\mathbb{R}$. Then there are a finite number of open sets $U_1, \ldots, U_n \in H$ with $[a, b] \subseteq U_1 \cup \cdots \cup U_n$.*

Problem 5. Prove this from Theorem 1 as follows. As S is closed and bounded there is a bounded closed interval $[a, b]$ with $S \subseteq [a, b]$. Let $U_0 = S^c$. Then U_0 is open. Let $H_0 = H \cup \{U_0\}$. Show that H_0 is an open cover of $[a, b]$ and that any finite subset of H_0 that covers $[a, b]$ contains a subset that covers S (just throw out S^c if it is a member of the subcover of $[a, b]$).