Math 554 Homework

Read Section 2.1 of the text. Be sure you know the following definition

Definition 1. We say that $f(x)$ approaches the limit L as x approaches x_0, and write

$$\lim_{x\to x_0} f(x) = L$$

if f is defined on some deleted neighborhood of x_0 and, for every $\varepsilon > 0$, there is a $\delta > 0$ such that

$$0 < |x - x_0| < \delta \qquad \Longrightarrow \qquad |f(x) - f(x_0)| < \varepsilon.$$

Here is an example if $f(x) = x^2 + x$ then

$$\lim_{x\to 3} x^2 + x = 12.$$

Scratch work:

$$|f(x) - 12| = |x^2 + x - 12| = |x - 3||x + 4|.$$

We would like this to be less than ε by making $|x - 3| < \delta$. The factor of $|x - 3|$ is good. To deal with the $|x + 4|$ factor, we assume that $\delta \le 1$. Then $|x-3| \le \delta \le 1$ implies $2 \le x \le 4$ and so $|x+4| \le 8$. Thus if we let $\delta = \min\{1, \varepsilon/8\}$ then things should work.

We now put aside our scratch work and become formal

Proposition 2. *If* $f(x) = x^2 + x$ *then*

$$\lim_{x\to 3} x^2 + x = 12.$$

Proof. Let

$$\delta = \min\left\{1, \frac{\varepsilon}{8}\right\}.$$

Then $|x - 3| < \delta$ implies.

$$|x + 4| = |(x - 3) + 7| \le |x - 3| + 7 < \delta + 7 \le 1 + 7 = 8.$$

Thus $0 < |x - 3| < \delta$ implies

$$\begin{aligned}
|f(x) - 12| &= |x^2 + x - 12| \\
&= |x - 3||x + 4| \\
&\le 8|x - 3| && (\text{as } |x + 4| < 8) \\
&< 8\left(\frac{\varepsilon}{8}\right) && (\text{as } |x - 3| < \delta \le \varepsilon/8) \\
&= \varepsilon.
\end{aligned}$$

□

Here are some for you to do. I am not so interested in seeing the scratch work, but I do want all the details of the proof put in. I am going to be very strict in grading these.

Problem 1. Show $\lim_{x\to -1} 3x^2 = 3$.

Here is an easier one

Problem 2. Let $a \neq 0$ and let $f(x) = ax + b$. Show $\lim_{x\to x_0} f(x) = ax_0 + b$. *Hint:* Let $\delta = \varepsilon/a$.

Here is another example. $\lim_{x\to 1} x^3 = 1$.

Scratch work: For $f(x) = x^3$ we have

$$|f(x) - 1| = |x^3 - 1| = |x^2 + x + 1||x - 1|.$$

We would like this to be less than ε by making $|x - 1| < \delta$. To deal with the $|x + 4|$ factor, we assume that $\delta \leq 1$. Then $|x - 1| \leq \delta \leq 1$ impies $0 < x < 2$ and so $|x^2 + x + 1| \leq 9$. So $\delta = \min\{1, \varepsilon/9\}$ should work.

Proposition 3. *If* $f(x) = x^3$*, then*

$$\lim_{x\to 1} x^3 = 1$$

Proof. Let

$$\delta = \min\{1, \varepsilon/9\}.$$

Then $0 < |x - 1| < \delta$ implies $0 < x < 2$ and therefore $|x^2 + x + 1| \leq 2^2 + 2 + 1 = 9$. Thus

$$\begin{aligned}
|f(x) - 1| &= |x^3 - 1| \\
&= |x^2 + x + 1||x - 1| \\
&\leq 9|x - 1| && (\text{as } |x^2 + x + 1| \leq 9) \\
&< 9\left(\frac{\varepsilon}{9}\right) && (\text{as } |x - 1| < \delta \leq \varepsilon/9) \\
&= \varepsilon.
\end{aligned}$$

□

Problem 3. Show $\lim_{x\to 3} 2x^3 = 54$.

Problem 4. Show $\lim_{x\to 1} \frac{1}{x} = 1$. *Hint:* $\frac{1}{x} - 1 = \frac{-(x-1)}{x}$.