Math 554 Homework

We need more practice with limits. Here is an example.

Proposition 1. *Let* $L \le f(x) \le g(x)$ *and assume*

$$\lim_{x\to x_0} g(x) = L.$$

Then

$$\lim_{x\to x_0} f(x) = L.$$

Proof. Let $\varepsilon > 0$. As $\lim_{x\to x_0} g(x) = L$ there is a $\delta > 0$ so that

$$0 < |x - x_0| < \delta \qquad \Longrightarrow \qquad |g(x) - L| < \varepsilon.$$

If $0 < |x - x_0| < \delta$ then, as $L \le f(x) \le g(x)$, we have

$$|f(x)-L| = f(x)-L, \qquad |g(x)-f(x)| = g(x)-f(x), \qquad |g(x)-L| = g(x)-L.$$

Thus $0 < |x - x_0| < \delta$ implies

$$|f(x) - L| = f(x) - L \le g(x) - L = |g(x) - L| < \varepsilon.$$

□

There is a more general form of this, often called the "squeeze lemma" in calculus books.

Proposition 2 (Squeeze Lemma)**.** *Let* f, g, *and* h *be defined in a punctured neighborhood of* x_0. *Assume*

$$g(x) \le f(x) \le h(x)$$

and

$$\lim_{x\to x_0} g(x) = \lim_{x\to x_0} h(x) = L.$$

then

$$\lim_{x\to x_0} f(x) = L.$$

Problem 1. Draw a picture and write a few sentences that make this look and sound reasonable.

Problem 2. Show $a \le b \le c$ implies $|b| \le \max\{|a|, |c|\}$.

Problem 3. Prove Proposition 2. *Hint:* $g(x) \le f(x) \le h(x)$ implies $g(x) - L \le f(x) - L \le h(x) - L$ and so by Problem 2 we have $|f(x) - L| \le \max\{|g(x) - L|, |h(x) - L|\}$. And we can make both of $|g(x) - L|$ and $|h(x) - L|$ small.

Problem 4. Show

$$\lim_{x\to 1} (x - 1)\sin(1/(x - 1)) = 0.$$

Hint: We know $\lim_{x\to 1}(x - 1) = \lim_{x\to 1}(-(x - 1)) = 0$ (you don't have to prove these). And $-(x-1) \le (x-1)\sin(1/(x-1)) \le (x-1)$ (explain why).

Read pages 37–40 on one sided limits in the text.

Problem 5. Do problems 7a and 7b on page 49 of the text.

Problem 6. Show that if f is defined in a punctured neighborhood of x_0 and $\lim_{x\to x_0} f(x) = L$, then the two one sided limits $\lim_{x\to x_0^-} f(x)$ and $\lim_{x\to x_0^+} f(x)$ both exist are are equal to L.

Problem 7. Show that if f is defined in a punctured neighborhood of x_0 and the two one sided limits $\lim_{x\to x_0^-} f(x)$ and $\lim_{x\to x_0^+} f(x)$ exist and have the same value L, then $\lim_{x\to x_0} f(x) = L$.

Putting these together we have:

Theorem 3. *Let f be defined on a punctured neighborhood of x_0. Show that* $\lim_{x\to x_0}$ *exists if and only if both the one side limits* $\lim_{x\to x_0^+} f(x)$ *and* $\lim_{x\to x_0^-} f(x)$ *exist and are equal.*