Math 554 Homework

Here is an important result we talked about in class, that the composition of continuous functions is continuous.

Theorem 1. *Let $g\colon (a,b) \to (A,B)$ be continuous and $f\colon (A,B) \to \mathbb{R}$ also be continuous. Then the composition $f \circ g\colon (a,b) \to \mathbb{R}$ is continuous. (The composition $f \circ g$ is the function defined by $(f \circ g)(x) = f(g(x))$.)*

Problem 1. Prove this along the following lines. Let $x_0 \in (a,b)$ and $\varepsilon > 0$.

(a) Explain why there here is a $\delta_1 > 0$ so that

$$|y - g(x_0)| < \delta_1 \qquad \Longrightarrow \qquad |f(y) - f(g(x_0))| < \varepsilon.$$

Hint: Any "solution" that has more than four sentences will be marked wrong. It is a definition.

(b) Explain why there is a $\delta > 0$ so that

$$|x - x_0| < \delta \qquad \Longrightarrow \qquad |g(x) - g(x_0)| < \delta_1$$

Hint: See the previous hint.

(c) Finish the proof.

Theorem 2. *Let $f\colon I \to \mathbb{R}$ be a continuous function on the interval I. If f is injective (i.e. one-to-one) then f is strictly monotone. (That is f is either strictly increasing or strictly decreasing.)*

Proof. We are not going to prove this. It is not hard, but after looking at several texts, the only proofs I could find evolved really annoying proofs by cases. (The lemma needed is that if f is injective, but not strictly monotone, then there are $x_1 < x_2 < x_3$ in the domain of f so that $f(x_2)$ is not between $f(x_1)$ and $f(x_3)$.) □

Remark. As we know that the square root function is continuous on $[0,\infty)$, that polynomials are continuous, and rational functions are continuous at points where the denominator is not zero, the last result implies the following are continuous

$$f(x) = \sqrt{5x^4 + 2x^2 + 3} \qquad \text{(the stuff in the } \sqrt{} \text{ is positive)}$$

$$g(x) = \frac{\sqrt{x^2+9}}{x^4+7}.$$

And once we know that cos is continuous we will have that

$$h(x) = \cos\left(\frac{x^2 + \cos^2(x)}{5 + \sqrt{2 + \cos(x)}}\right)$$

is continuous. And of course we can build up much more complicated functions by repeated use of function composition.

Proposition 3. *Let $f\colon [a,b] \to [A,B]$ be continuous and strictly increasing. Then the inverse $f^{-1}\colon [A,B] \to [a,b]$ is also strictly increasing. (A corresponding statement is true for continuous strictly decreasing functions.)*

Problem 2. Prove this.

Problem 3. Let $f\colon (a,b)\to (A,B)$ be continuous and onto (i.e. surjective) and strictly increasing. Show that for any $y_0\in (A,B)$ that

$$\lim_{y\to y_0^-} f^{-1}(y) = f^{-1}(y_0).$$

Hint: From Proposition 3 the function f^{-1} is increasing. Therefore

$$\alpha = \lim_{y\to y_0^-} f^{-1}(y)$$

exits from a result you proved on the last homework. Assume, towards a contradiction, that $\alpha \neq f^{-1}(y_0)$.

(a) Then $\alpha < f^{-1}(y_0)$. *Hint:* As f^{-1} is increasing, if $y < y_0$, then $f^{-1}(y) < f^{-1}(y_0)$.
(b) Show if $y < y_0$, then $f^{-1}(y) \le \alpha$.
(c) As f is increasing and $\alpha < f^{-1}(y_0)$ we have $f(\alpha) < f(f^{-1}(y_0)) = y_0$. Let y_1 be so that $f(\alpha) < y_1 < y_0$. Explain why there is an x_1 with $f(x_1) = y_1$. Thus $f^{-1}(y_1) = x_1$.
(d) Explain why (c) contradicts (b) which completes the proof. *Hint:* Show $f^{-1}(y_1) > \alpha$.