Math 554 Homework, Some Solutions.

Proposition 1 (Squeeze Lemma)**.** *Let f, g, and h be defined in a punctured neighborhood of x_0. Assume*

$$g(x) \le f(x) \le h(x)$$

and

$$\lim_{x\to x_0} g(x) = \lim_{x\to x_0} h(x) = L.$$

then

$$\lim_{x\to x_0} f(x) = L.$$

Problem 1. Draw a picture and write a few sentences that make this look and sound reasonable.

Problem 2. Show $a \le b \le c$ implies $|b| \le \max\{|a|, |c|\}$.

Solution. This is just an annoying proof by cases.

Case 1. $0 \le b$. Then $0 \le b \le c$ so that $c \ge 0$. Therefore

$$|b| = b \le c = |c| \le \max\{|a|, |c|\}.$$

Case 2. $b < 0$. Then $a \le b \le 0$. Thus $0 \le -b \le -a$. Whence

$$|b| = -b \le -a = |a| \le \max\{|a|, |c|\}.$$

As either $b \ge 0$ or $b < 0$ this covers all cases. □

Problem 3. Prove Proposition 1. *Hint:* $g(x) \le f(x) \le h(x)$ implies $g(x) - L \le f(x) - L \le h(x) - L$ and so by Problem 2 we have $|f(x) - L| \le \max\{|g(x) - L|, |h(x) - L|\}$. And we can make both of $|g(x) - L|$ and $|h(x) - L|$ small.

Solution. Let $\varepsilon > 0$. As in the hint we have

$$|f(x) - L| \le \max\{|g(x) - L|, |h(x) - L|\} \tag{1}$$

As $\lim_{x\to x_0} g(x) = \lim_{x\to x_0} h(x) = L$ there are $\delta_1 > 0$ and $\delta_2 > 0$ so that

$$0 < |x - x_0| < \delta_1 \quad \Longrightarrow \quad |g(x) - L| < \varepsilon$$
$$0 < |x - x_0| < \delta_2 \quad \Longrightarrow \quad |h(x) - L| < \varepsilon$$

Let $\delta = \min\{\delta_1, \delta_2\}$. Then $0 < |x - x_0| < \delta$, along with the inequality (1), implies

$$|f(x) - L| \le \max\{|g(x) - L|, |h(x) - L|\} < \max\{\varepsilon, \varepsilon\} = \varepsilon,$$

as required to show $\lim_{x\to x_0} f(x) = L$. □

Problem 4. Show

$$\lim_{x\to 1} (x - 1)\sin(1/(x - 1)) = 0.$$

Hint: We know $\lim_{x\to 1} |x - 1| = \lim_{x\to 1}(-|x - 1|) = 0$ (you don't have to prove these). And $-|x - 1| \le (x - 1)\sin(1/(x - 1)) \le |x - 1|$ (explain why).

Solution. Done in class. □

Read pages 37–40 on one sided limits in the text.

Problem 5. Do problems 7a and 7b on page 49 of the text.

Solution to 7a. Let $f(x) = (x + |x|)/x$. Then

$$f(x) = \begin{cases} \dfrac{x+x}{x} = 2, & x > 0; \\ \\ \dfrac{x-x}{x} = 0, & x < 0 \end{cases}$$

Thus

$$\lim_{x\to 0^-} f(x) = 0, \qquad \text{and} \qquad \lim_{x\to 0^+} f(x) = 2.$$

To prove this let $\varepsilon > 0$ and let $\delta = 17$. Then

$$-\delta < x < 0 \qquad \Longrightarrow \qquad |f(x) - 0| = |0 - 0| = 0 < \varepsilon$$

which proves $\lim_{x\to 0^-} f(x) = 0$ and

$$0 < x < \delta \qquad \Longrightarrow \qquad |f(x) - 2| = |2 - 2| = 0 < \varepsilon$$

which proves $\lim_{x\to 0^+} f(x) = 2$.

There was nothing special about $\delta = 17$, any positive number would have worked on this function. The reason for this is that f is constant on each of the intervals $(-\infty, 0)$ and $(0, \infty)$. □

Solution to 7b. Let $f(x) = x\cos(1/x) + \sin(1/x) + \sin(1/|x|)$. We can simplify this a bit. If $x > 0$, then $|x| = x$ and we have

$$f(x) = x\cos(1/x) + \sin(1/x) + \sin(1/|x|) = x\cos(1/x) + 2\sin(1/x).$$

and if $x < 0$ then $|x| = -x$ and so $\sin(1/|x|) = \sin(1/(-x)) = -\sin(1/x)$ and thus for $x < 0$

$$f(x) = x\cos(1/x)+\sin(1/x)+\sin(1/|x|) = x\cos(1/x)+\sin(1/x)-\sin(1/x) = x\cos(1/x).$$

In summary

$$f(x) = \begin{cases} x\cos(1/x) + 2\sin(1/x), & x > 0; \\ x\cos(1/x), & x < 0. \end{cases}$$

Thus

$$\lim_{x\to 0^+} f(x) \quad \text{Does not exist.}$$

and

$$\lim_{x\to 0^-} f(x) = 0.$$

We are only required to the ε, δ stuff for $\lim_{x\to 0^-} f(x) = 0$. So let $\varepsilon > 0$ and let $\delta = \varepsilon$. Then

$$-\delta < x < 0 \qquad \Longrightarrow \qquad |f(x) - 0| = |x\cos(1/x)| \le |x| < \delta = \varepsilon,$$

where we have used that $-\delta < x < 0$ implies $|x| < \delta$ and $|\cos(1/x)| \le 1$. □

Problem 6. Show that if f is defined in a punctured neighborhood of x_0 and $\lim_{x\to x_0} f(x) = L$, then the two one sided limits $\lim_{x\to x_0^-} f(x)$ and $\lim_{x\to x_0^+} f(x)$ both exist are are equal to L.

Solution. Let $\varepsilon > 0$. Then, as $\lim_{x\to x_0} f(x) = L$, there is a $\delta > 0$ so that

$$0 < |x - x_0| < \delta \quad \Longrightarrow \quad |f(x) - L| < \varepsilon. \tag{2}$$

As $x_0 < x < x_0 + \delta$ implies $0 < |x - x_0| < \delta$ the implication (2) yields

$$x_0 < x < x_0 + \delta \quad \Longrightarrow \quad |f(x) - L| < \varepsilon,$$

which is what is required to show $\lim_{x\to x_0^+} f(x) = L$.

Likewise $x_0 - \delta < x < x_0$ implies $0 < |x - x_0| < \varepsilon$ and we can again use (2) get

$$x_0 - \delta < x < x_0 \quad \Longrightarrow \quad |f(x) - L| < \varepsilon,$$

which shows $\lim_{x\to x_0^-} f(x) = L$. □

Problem 7. Show that if f is defined in a punctured neighborhood of x_0 and the two one sided limits $\lim_{x\to x_0^-} f(x)$ and $\lim_{x\to x_0^+} f(x)$ exist and have the same value L, then $\lim_{x\to x_0} f(x) = L$.

Solution. Let $\varepsilon > 0$. As $\lim_{x\to x_0^-} f(x) = L$, there is a $\delta_1 > 0$ so that

$$x_0 - \delta_1 < x < x_0 \quad \Longrightarrow \quad |f(x) - L| < \varepsilon. \tag{3}$$

As $\lim_{x\to x_0^-} f(x) = L$, there is a $\delta_2 > 0$ so that

$$x_0 < x < x_0 + \delta_2 \quad \Longrightarrow \quad |f(x) - L| < \varepsilon. \tag{4}$$

Let $\delta = \min\{\delta_1, \delta_2\}$. If $0 < |x - x_0| < \delta$ then one of two cases holds:

Case 1. $x < x_0$, in which case $x_0 - \delta < x < x_0$. But $\delta \leq \delta_1$ thus $x_0 - \delta_1 < x < x_0$ holds and so by (3) $|f(x) - L| < \varepsilon$.

Case 2. $x_0 < x$, in which case $x_0 < x < x_0 + \delta$. But $\delta \leq \delta_2$ thus $x_0 < x < x_0 + \delta_2$ holds and so by (4) $|f(x) - L| < \varepsilon$.

Putting the cases together we see

$$0 < |x - x_0| < \delta \quad \Longrightarrow \quad |f(x) - L| < \varepsilon$$

and we have thus shown $\lim_{x\to x_0} f(x) = L$. □

Putting these together we have:

Theorem 2. *Let f be defined on a punctured neighborhood of x_0. Show that $\lim_{x\to x_0}$ exists if and only if both the one side limits $\lim_{x\to x_0^+} f(x)$ and $\lim_{x\to x_0^-} f(x)$ exist and are equal.*