Math 554, Test 1, Take Home Portion.

Solutions:

Definition 1. A subset, S, of the real numbers, $\mathbb{R}$, is ***compact*** iff every open cover, $\mathcal{H}$, of S has a finite sub-cover.

With this terminology the Heine-Borel Theorem can be stated as

Theorem 2. *Every closed bounded subset of* $\mathbb{R}$ *is compact.*

We wish to show the converse of this, that is that every compact subset of $\mathbb{R}$ is closed and bounded. We split this into two parts. But first we recall

Archimedean Principle. *Let* ε *and* p *be positive numbers. Then there is a positive integer* n *such that* $n\varepsilon > p$. □

Problem 1. If S is compact, then S is bounded. *Hint:* Let S be compact and let $\mathcal{H} = \{(-n, n) : n = 1, 2, 3, \ldots\}$.

(a) Show $\mathcal{H}$ is an open cover of S. (This should involving using the Archimedean Principle.)
(b) Use that $\mathcal{H}$ has a finite sub-cover to show S is bounded.

Proof of (a). The elements of $\mathcal{H}$ are open intervals, and so are open. So we just need to show that $\mathcal{H}$ is a cover of S. In fact we will show that $\mathcal{H}$ is a cover of all of $\mathbb{R}$, therefore a cover of any subset of $\mathbb{R}$. To show that $\mathcal{H}$ covers $\mathbb{R}$ we need to show that for any $x \in \mathbb{R}$ there is an element $(-n, n) \in \mathcal{H}$ with $x \in (-n, n)$.

So let $x \in \mathbb{R}$. By the Archimedian principle (with $\varepsilon = 1$ and $p = |x|$) there is a positive integer n such that $|x| < n$. But then $x \in (-n, n)$. This is what we needed to show that $\mathcal{H}$ is a cover of $\mathbb{R}$. □

Proof of (b). We are assuming that every open cover of S has a finite sub-cover. Applying this to the open cover $\mathcal{H}$ of part (a) we get a finite subset

$$\{(-n_1, n_1), (-n_2, n_2), \ldots, (-n_k, n_k)\} \subseteq \mathcal{H}$$

that covers S. That is

$$S \subset (-n_1, n_1) \cup (-n_2, n_2) \cup \cdots \cup (-n_k, n_k).$$

Let $n = \max\{n_1, n_2, \ldots, n_k\}$. Then

$$(-n_1, n_1) \cup (-n_2, n_2) \cup \cdots \cup (-n_k, n_k) = (-n, n).$$

Thus

$$S \subset (-n_1, n_1) \cup (-n_2, n_2) \cup \cdots \cup (-n_k, n_k) = (-n, n).$$

Thus for this n, if $x \in S$ we have $-n < x < n$. This shows that S is bounded both from above and below. Thus S is bounded. □

Problem 2. If S is compact, then S is closed. *Hint:* Use a proof by contradiction. Assume that S is compact, but not closed. Then S has a limit point x_0 with $x_0 \notin S$. For each positive integer n let

$$U_n = (-\infty, x_0 - 1/n) \cup (x_0 + 1/n, \infty)$$

and let

$$\mathcal{H} = \{U_n : n = 1, 2, 3, \ldots\}$$

(a) Show that $\mathcal{H}$ is an open cover of S. (This will use that $x_0 \notin S$ and the Archimedean Principle).
(b) Use that $\mathcal{H}$ has a finite sub-cover and that x_0 is a limit point to get a contradiction.

Proof of (a). Each of the sets U_n is the union of two open intervals and thus U_n is open. Thus $\mathcal{H}$ is colloction of open sets. For future use note that U_n is just the compliment of the closed interval $[x_0 - 1/n, x_0 + 1/n]$, that is

$$U_n = [x_0 - 1/n, x_0 + 1/n]^c.$$

We now need to show that $\mathcal{H}$ covers S. Let $x \in S$. As $x_0 \notin S$ we see that $x \neq x_0$. Thus $x < x_0$ or $x > x_0$.

Case 1. $x < x_0$. Then $x_0 - x > 0$ and so by the Archimedian Principle (with $\varepsilon = (x_0 - x)$ and $p = 1$)[1] there is a positive integer n with $n(x_0 - x) > 1$. Then

$$n(x_0 - x) > 1 \quad \Longrightarrow \quad (x_0 - x) > 1/n \quad \Longrightarrow \quad x_0 - 1/n > x.$$

Thus $x \in (-\infty, x_0 - 1/n) \subseteq U_n$. Whence for any $x < x_0$ there is an n such that $x \in U_n$.

Case 2. $x > x_0$. Then $x - x_0 > 0$ and so by the Archimedian Principle (with $\varepsilon = (x - x_0)$ and $p = 1$) there is a positive integer n such that $1 < (x - x_0)$. Then

$$n(x - x_0) > 1 \quad \Longrightarrow \quad (x - x_0) > 1/n \quad \Longrightarrow \quad x > x_0 + 1/n.$$

Thus $x \in (x_0 + 1/n, \infty) \subseteq U_n$. Whence for any $x < x_0$ there is an n such that $x \in U_n$.

Combining these two cases shows that if $x \in S$, then there is an n so that $x \in U_n$. Therefore $\mathcal{H} = \{U_n : n = 1, 2, \ldots\}$ is an open cover of S. □

Remark. It is possible to both of the two cases above all at once as follows. If $x \in S$, then $x \neq x_0$. Thus $|x - x_0| > 0$ and by the Archimedean Principle there is a positive integer n with $n|x - x_0| > 1$. Then $|x - x_0| > 1/n$ and so $x \notin [x_0 - 1/n, x_0 + 1/n]$. That is $x \in [x_0 - 1/n, x_0 + 1/n]^c = U_n$.

Proof of (b). We are assuming that x_0 is a limit point of S and therefore every punctured neighborhood of x_0 contains a point of S.

As S is compact the opne cover $\mathcal{H}$ of S contains a finite sub-cover

$$\widetilde{\mathcal{H}} = \{U_{n_1}, U_{n_2}, \ldots, U_{n_k}\}$$

[1] Here is the scratch work to arrive at this choice of n. We want $x < x_0 - 1/n$. Rearranging this is the same as $1/n < x_0 - x$, which is equivalent to $1 < n(x_0 - x)$.

that covers S. Let $n = \max\{n_1, n_2, \dots, n_k\}$. Then as $\widetilde{\mathcal{H}}$ covers S we have

$$S \subseteq U_{n_1} \cup U_{n_2} \cup \cdots \cup U_{n_k} = U_n.$$

As U_n is the compliment of $[x_0 - 1/n, x_0 + 1/n]$ that $\subseteq U_n$ implies that S is disjoint from $[x_0 - 1/n, x_0 + 1/n]$ and therefore S contains no point of the punctured neighborhood $\{x : 0 < |x - x_0| < 1/n\}$ of x_0. This contradicts that x_0 is a limit point of S and completes the proof. □

Problem 3. Let f and g be defined in a deleted neighborhood of x_0. Assume that f is bounded, say $|f(x)| \leq A$ for all x in the domain of f, and that $\lim_{x\to x_0} g(x) = 0$. Show

$$\lim_{x\to x_0} f(x)g(x) = 0.$$

Remark. Note that we are *not* assuming that $\lim_{x\to x_0} f(x)$ exists.

Proof. Let $\varepsilon > 0$. Then as $\lim_{x\to x_0} g(x) = 0$, there is a $\delta > 0$ such that

$$0 < |x - x_0| < \delta \qquad \Longrightarrow \qquad |g(x) - 0| < \frac{\varepsilon}{A+1}.$$

Thus if $0 < |x - x_0| < \delta$ we have

$$\begin{aligned} |f(x)g(x) - 0| &= |f(x)||g(x)| \\ &\leq A|g(x)| && \text{(as } |f(x)| \leq A) \\ &< A\left(\frac{\varepsilon}{A+1}\right) && \text{(as } |g(x)| = |g(x) - 0| < \frac{\varepsilon}{A+1}\ ) \\ &< \varepsilon. \end{aligned}$$

That is we have just shown

$$0 < |x - x_0| < \delta \qquad \Longrightarrow \qquad |f(x)g(x) - 0| < \varepsilon.$$

Therefore $\lim_{x\to x_0} f(x)g(x) = 0$. □