Math 554, Test 3 — Name: Answer Key

The problems are 20 points each.

1. (a) State the definition of a function f being ***differentiable*** at x_0.

Solution: The function f is ***differentiable*** at x_0 iff the limit

$$f'(x_0) = \lim_{x\to x_0} \frac{f(x) - f(x_0)}{x - x_0}$$

exists. □

(b) Let f be defined by $f(x) = \begin{cases} x^2\cos(1/x^3), & x \neq 0; \\ 0, & x = 0. \end{cases}$
Show directly from the definition that f is differentiable at 0.

Solution: We wish to show the existence of the limit

$$\lim_{x\to 0} \frac{f(x) - f(0)}{x - 0} = \lim_{x\to 0} \frac{x^2\cos(1/x^3)}{x} = x\cos(1/x^3).$$

Let $\varepsilon > 0$ and let $\delta = \varepsilon$. Then $0 < |x - 0| < \delta$ implies

$$\left| \frac{f(x) - f(0)}{x - 0} - 0 \right| = \left| x\cos(1/x^3) \right| \leq |x| = |x - 0| < \delta = \varepsilon.$$

Thus $f'(0) = \lim_{x\to 0} \frac{f(x)-f(0)}{x-0}$ exists and has the value $f'(0) = 0$. □

Alternate Solution: You could also do this by noting

$$-|x| \leq \frac{f(x) - f(0)}{x - 0} = x\cos(1/x^3) \leq |x|$$

and using the squeeze lemma. □

(c) Let f and g be differentiable at x_0. Prove from the definition and known properties of limits that the product $p(x) = f(x)g(x)$ is differentiable at x_0.

Solution: We are given that the limits

$$f'(x_0) = \lim_{x\to x_0} \frac{f(x) - f(x_0)}{x - x_0}, \qquad g'(x_0) = \lim_{x\to x_0} \frac{g(x) - g(x_0)}{x - x_0}$$

exist. We also know that as g is differentiable at x_0 it is also continuous at x_0. Thus the limit

$$\lim_{x\to x_0} g(x) = g(x_0)$$

exists.

To show that $p'(x_0)$ exists we need to show the existence of the limit as $x \to x_0$ of

$$\begin{aligned}\frac{f(x)g(x) - f(x_0)g(x_0)}{x - x_0} &= \frac{f(x)g(x) - f(x_0)g(x) + f(x_0)g(x) - f(x_0)g(x_0)}{x - x_0} \\ &= \frac{f(x) - f(x_0)}{x - x_0} g(x) + f(x_0)\frac{g(x) - g(x_0)}{x - x_0}\end{aligned}$$

Now using basic properties of limits can show the limit for for $p'(x_0)$ exists as follows

$$\begin{aligned}\lim_{x\to x_0} \frac{p(x) - p(x_0)}{x - x_0} &= \lim_{x\to x_0} \left(\frac{f(x) - f(x_0)}{x - x_0} g(x) + f(x_0)\frac{g(x) - g(x_0)}{x - x_0} \right) \\ &= \lim_{x\to x_0} \frac{f(x) - f(x_0)}{x - x_0} \lim_{x\to x_0} g(x) + f(x_0) \lim_{x\to x_0} \frac{g(x) - g(x_0)}{x - x_0} \\ &= f'(x_0)g(x_0) + f(x_0)g'(x_0)\end{aligned}$$

as required. □

2. (a) State the ***Mean Value Theorem***.

Solution: Let f be differentiable on the open interval (a, b) and continuous on the closed interval $[a, b]$. Then there is a ξ between a and b such that

$$f(b) - f(a) = f'(\xi)(b - a).$$

□

(b) Use the mean value theorem to show that if f is differentiable on an interval I and $f'(x) = 0$ for all $x \in I$, then f is a constant. □

Solution: Let $a \in I$ then for any point other point $b \in I$ we can use the mean value theorem to find a ξ between a and b such that

$$f(b) - f(a) = f'(\xi)(b - a) = 0(b - a) = 0.$$

Therefore $f(b) = f(a)$ for all $b \in I$. That is f has the constant value $f(a)$ on I. □

(c) Show for any real numbers a and b that $\left|(2a + a^3) - (2b + b^3)\right| \geq 2|a - b|$.

Solution: Let f be defined on $\mathbb{R}$ by $f(x) = 2x + x^3$. Then we are to show for any $a, b \in \mathbb{R}$ that

$$|f(a) - f(b)| \geq 2|b - a|.$$

Note f is differentiable, with derivative $f'(x) = 2 + 3x^2$, on $\mathbb{R}$ and therefore there is a ξ between a and b such that $f(b) - f(a) = f'(\xi)(b - a)$. Thus

$$|f(a) - f(b)| = |f'(\xi)(a - b)| = |2 + 3\xi^2||a - b| \geq 2|b - a|,$$

as $2 + 2\xi^2 \geq 2$ because $\xi^2 \geq 0$. □

3. Let f be defined in a neighborhood of x_0 and assume there is a function E defined on a neighborhood of x_0 such that for some constant m

$$f(x) = m(x - x_0) + E(x)(x - x_0)$$

and

$$\lim_{x \to x_0} E(x) = 0.$$

Show f is differentiable at x_0 and that $f'(x_0) = m$.

Solution: First note

$$f(x_0) = m(x_0 - x_0) + E(x_0)(x_0 - x_0) = 0$$

Thus we have

$$\begin{aligned}
\lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0} &= \lim_{x \to x_0} \frac{f(x) - 0}{x - x_0} \\
&= \lim_{x \to x_0} \frac{m(x - x_0) + E(x)(x - x_0)}{x - x_0} \\
&= \lim_{x \to x_0} (m + E(x)) \\
&= m + 0 \\
&= m.
\end{aligned}$$

and so $f'(x_0)$ exists and $f'(x_0) = m$. $\square$

Alternate Solution: During the test the assumption on f was changed to

$$f(x) - f(x_0) = m(x - x_0) + E(x)(x - x_0)$$

and

$$\lim_{x \to x_0} E(x) = 0,$$

in which case solution is a bit shorter.

$$\begin{aligned}
\lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0} &= \lim_{x \to x_0} \frac{m(x - x_0) + E(x)(x - x_0)}{x - x_0} \\
&= \lim_{x \to x_0} (m + E(x)) \\
&= m + 0 \\
&= m.
\end{aligned}$$

and so $f'(x_0)$ exists and $f'(x_0) = m$. $\square$

4. Let h be a twice differentiable function on an open interval I. Assume that there are distinct points $x_0, x_1, x_2 \in I$ with $x_0 < x_1 < x_2$ and $h(x_0) = h(x_1) = h(x_2) = 0$.

(a) Show there is a $\xi \in (x_0, x_2)$ such that $h''(\xi) = 0$.

Solution: By two applications of Rolle's theorem to the differentiable function h, once on the interval $[x_0, x_1]$ and once to the interval $[x_1, x_2]$, we find there exists $\xi_1 \in (x_0, x_1)$ and $\xi_2 \in (x_1, x_2)$ with

$$h'(\xi_1) = h'(\xi_2) = 0.$$

Another application of Rolle's Theorem, this time to the differentiable function h' on the interval $[\xi_1, \xi_2]$, we find there is a $\xi \in (\xi_1, \xi_2)$ with

$$(h')'(\xi) = h''(\xi) = 0$$

and $x_1 < \xi_1 < \xi < \xi_2 < x_2$ so $\xi \in (x_0, x_1)$. □

(b) Show that if f and g are three time differentiable functions on I and $f(x_j) = g(x_j)$ for $j = 0, 1, 2$ then there is a $\xi \in (x_0, x_2)$ such that $f''(\xi) = g''(\xi)$.

Solution: Let $h = f - g$. Then h is twice differentiable and $h(x_j) = f(x_j) - g(x_j) = 0$ for $j = 0, 1, 2$. Therefore by part (a) there is a ξ between x_0 and x_2 with $h''(\xi) = f''(\xi) - g''(\xi) = 0$. Thus $f''(\xi) = g''(\xi) = 0$. □

5. Let I be an open interval and f a two times differentiable function on I.

(a) State Taylor's theorem with Lagrange's form of the remainder for f. (That is the form of Taylor's that has the remainder term with a second derivative in it.)

Solution: For any $x_0, x \in I$ there is a ξ between x_0 and x such that

$$f(x) = f(x_0) + f'(x_0)(x - x_0) + f''(\xi)\frac{(x - x_0)^2}{2}.$$

□

(b) Show that if f is two times differentiable on I and $f''(x) \geq 0$ for $x \in I$ that if $f'(x_0) = 0$ for some $x_0 \in I$ then x_0 is a minimizer of f on I. (That is $f(x_0) \leq f(x)$ for all $x \in I$). *Hint:* This should follow at once from Taylor's Theorem and the fact that squares of real numbers are positive.

Solution: Let $x \in I$. Then as $f'(x_0) = 0$ we have from Taylor's theorem

$$\begin{aligned} f(x) &= f(x_0) + f'(x_0)(x - x_0) + f''(\xi)\frac{(x - x_0)^2}{2} \\ &= f(x_0) + 0 + f''(\xi)\frac{(x - x_0)^2}{2} \\ &\geq f(x_0) \end{aligned}$$

as $f''(\xi) \geq 0$ and $(x - x_0)^2 \geq 0$. □