Here is Polina's solution to the problem of showing that $f(x) = \sqrt{x}$ is uniformly on $[0, 1]$. Her argument shows something a bit more general. As with our other proof of this result it is based on the algebraic identity

$$\sqrt{x} - \sqrt{y} = \frac{x - y}{\sqrt{x} + \sqrt{y}} \tag{1}$$

**Proposition 1.** *The function $f \colon [0, \infty) \to \mathbb{R}$ given by $f(x) = \sqrt{x}$ is uniformly continuous. (This is more general is we have replaced $[0, 1]$ with the larger set $[0, \infty)$.)*

*Proof.* Let $\varepsilon > 0$ and let $\delta = \varepsilon^2$. We wish to show that for for all $x, y \in [0, \infty)$

$$|x - y| < \delta \qquad \Longrightarrow \qquad |f(x) - f(y)| = |\sqrt{x} - \sqrt{y}| < \varepsilon. \tag{2}$$

There are two cases

*Case 1.* $\sqrt{x} + \sqrt{y} < \varepsilon$. Then by the triangle inequality

$$|\sqrt{x} - \sqrt{y}| \leq \sqrt{x} + \sqrt{y} < \varepsilon.$$

*Case 2.* $\sqrt{x} + \sqrt{y} \geq \varepsilon$. Use equation (1)

$$\begin{aligned} |\sqrt{x} - \sqrt{y}| &= \frac{|x - y|}{\sqrt{x} + \sqrt{y}} \\ &\leq \frac{|x - y|}{\varepsilon} && \text{(The denominator is smaller)} \\ &< \frac{\varepsilon^2}{\varepsilon} && \text{(as } |x - y| < \delta = \varepsilon^2\text{)} \\ &= \varepsilon. \end{aligned}$$

This covers all cases and we are are done. □

And just to be complete here as an argument for the same result for the $n$-th root function. It is not as nice an argument, and it would be interesting to find a more intuitive proof.

**Proposition 2.** *Let $n$ be a positive integer. Then the function $f \colon [0, \infty) \to \mathbb{R}$ given by*

$$f(x) = x^{1/n}$$

*is uniformly continuous on $[0, \infty)$.*

*Proof.* Again this will be based on an algebraic identity. Recall that

$$a^n - b^n = (a - b)(a^{n-1} + a^{n-2}b + \cdots + b^{n-1}).$$

Let $a = y^{1/n}$ and $b = x^{1/n}$. Then this becomes

$$y - x = \left(y^{1/n} - x^{1/n}\right)\left(y^{(n-1)/n} + y^{(n-2)/n}x^{1/n} + \cdots + x^{(n-1)/n}\right).$$

Divide this by $\left(x^{(n-1)/n} + x^{(n-2)/n}y^{1/n} + \cdots + y^{(n-1)/n}\right)$ to get

$$y^{1/n} - x^{1/n} = \frac{y - x}{y^{(n-1)/n} + y^{(n-2)/n}x^{1/n} + \cdots + x^{(n-1)/n}}$$

In what follows we want to make use of the special case where $y = x + \delta$ in which case we get

$$
\begin{aligned}
(x+\delta)^{1/n} &= \frac{x+\delta-x}{(x+\delta)^{(n-1)/n} + (x+\delta)^{(n-2)/n}x^{1/n} + \cdots + x^{1/n}} \\
&= \frac{\delta}{(x+\delta)^{(n-1)/n} + \cdots + x^{1/n}}
\end{aligned}
$$

Let $\varepsilon > 0$ and $\delta = \varepsilon^n$. Let $x, y \geq 0$ with $|x-y| < \delta$. Without loss of generality we can assume that $x \leq y$. This implies that $x \leq y < x + \delta$.

$$
\begin{aligned}
|x^{1/n} - y^{1/n}| &= y^{1/n} - x^{1/n} && (\text{as } y^{1/n} \geq x^{1/n}) \\
&< (x+\delta)^{1/n} - x^{1/n} && (\text{as } y^{1/n} < (x+\delta)^{1/n}) \\
&= \frac{\delta}{(x+\delta)^{(n-1)/n} + \cdots + x^{1/n}} && (\text{the identity above}) \\
&\leq \frac{\delta}{(0+\delta)^{(n-1)/n} + \cdots + 0^{1/n}} && (x = 0 \text{ makes the denominator smaller}) \\
&= \frac{\delta}{\delta^{(n-1)/n}} \\
&= \delta^{1/n} \\
&= (\varepsilon^n)^{1/n} && (\text{as } \delta = \varepsilon^n) \\
&= \varepsilon.
\end{aligned}
$$

Thus we have shown that if $x, y \in [0, \infty)$ and $\delta = \varepsilon^n$ that

$$|x - y| < \delta \qquad \Longrightarrow \qquad |x^{1/n} - y^{1/n}| < \varepsilon.$$

Thus $f(x) = x^{1/n}$ is uniformly continuous on $[0, \infty)$. $\square$