Mathematics 241 Test #2 Name: Key

Show your work to get credit. An answer with no work will not get credit.

1. (25 points) Compute the following partial derivatives.

(a) $f = xy^2z^3 + e^{2x+yz^2}$

$f_x = y^2z^3 + 2e^{2x+yz^2}$

$f_y = 2xyz^3 + z^2 e^{2x+yz^2}$

$f_z = 3xy^2z^2 + 2yz\, e^{2x+yz^2}$

(b) $g = r^3\cos(\theta)$

$g_\theta = -r^3 \sin\theta$

$g_r = 3r^2 \sin\theta$

$g_{\theta\theta} = -r^3 \cos\theta$

$g_{r\theta} = -3r^2 \sin\theta$

$g_{rr\theta} = -6r \sin\theta$

2. (6 points) At what point does the line $\mathbf{r}(t) = \langle 1+2t, 3-2t, 4+t\rangle$ intersect the plane $x+y+z = 6$?

Point of intersection is $\langle -3, 7, 2$

On the line

$x = 1+2t,\ y = 3-2t,\ z = 4+t.$

Plug these into the equation for the plane

$(1+2t) + (3-2t) + (4+t) = 6$

$8 + t = 6$

$t = -2$

so the point of intersection is

$\vec{r}(-2) = \langle 1-4, 3+4, 4-2\rangle = \langle -3, 7, 2\rangle$

3. (7 points) Find both the tangent vector $\mathbf{r}'$ and the unit tangent vector, $\mathbf{t}$, to the curve

$$\mathbf{r}(t) = t^2\mathbf{i} + t^3\mathbf{j}$$

at the point where $t = 1$.

$\mathbf{r}'(1) =$ $2\hat{i} + 3\hat{j} = \langle 2, 3 \rangle$

$\vec{r}\,'(t) = 2t\hat{i} + 3t\hat{j}$

when $t = 1$.

$\vec{r}\,'(1) = 2\hat{i} + 3\hat{j}$

$\vec{T}(1) = \dfrac{\vec{r}\,'(1)}{|\vec{r}\,'(1)|} = \dfrac{2\hat{i}+3\hat{j}}{\sqrt{4+9}}$

$\mathbf{t}(1) =$ $\frac{2}{\sqrt{13}}\hat{i} + \frac{3}{\sqrt{13}}\hat{j}$

4. (7 points) Find the parametric form of the tangent line to $\mathbf{r}(t) = \langle 1+t, t^2, 3-2t \rangle$ at the point where $t = -1$.

$x = 0 + t,\ y = 1 - 2t,\ z = 5 - 2t$

This is the line through $\vec{r}(-1) = \langle 0, 1, 5 \rangle$ in the direction of $\vec{v} = \vec{r}\,'(-1)$.

$\vec{r}\,'(t) = \langle 1, 2t, -2 \rangle$

$\vec{r}\,'(-1) = \langle 1, -2, -2 \rangle$

5. (7 points) For $\mathbf{u}(t) = \langle 6t^2, 4t, 3-2t \rangle$ compute $\int_{-1}^{2} \mathbf{u}(t)\, dt$.

$= \int_{-1}^{2} \langle 6t^2, 4t, 3-2t \rangle\, dt$

$\int_{-1}^{2} \mathbf{u}(t)\, dt =$ $\langle 18, 6, 6 \rangle$

$= \langle 2t^3, 2t^2, 3t - t^2 \rangle \Big|_{-1}^{2}$

$= \langle 2(2)^3, 2(2)^2, 3(2) - (2)^2 \rangle - \langle 2(-1)^3, 2(-1)^2, 3(-1) - (-1)^2 \rangle$

$= \langle 16, 4, 2 \rangle - \langle -2, -2, 4 \rangle$

$= \langle 18, 6, 6 \rangle$

6. (7 points) For

$$\mathbf{r}(t) = e^t\mathbf{i} + (2t+1)\mathbf{j} + t^3\mathbf{k}$$

set up the integral for the length of $\mathbf{r}$ between $t = 1$ and $t = 4$. (Note that you only have to set up the integral, not evaluate it.)

The integral for the length is $\int_1^4 \sqrt{e^{2t} + 4 + 9t^4}\,dt$

$$L = \int_1^4 |\vec{r}\,'(t)|\,dt = \int_1^4 \sqrt{\langle e^t, 2, 3t^2\rangle}\,dt$$

$$= \int_1^4 \sqrt{e^{2t} + 4 + 9t^4}\,dt$$

7. (7.points) What is the acceleration vector $\mathbf{a}(t)$ for a particle that has position $\mathbf{r}(t) = \langle e^t + t, 3\sin(t)\rangle$?

$\mathbf{a}(t) = \langle e^t, -3\sin t\rangle$

$\vec{v}(t) =$ velocity $= \vec{r}\,'(t)$

$= \langle e^t + 1, 3\cos t\rangle$

$\vec{a}(t) = \vec{v}\,'(t) = \langle e^t, -3\sin t\rangle$

8. (10 points) If $f(1,2) = 3$, $f_x(1,2) = -1$, and $f_y(1,2) = 4$ then

(a) what is the equation of the tangent plane to the graph $z = f(x,y)$ at the point where $(x,y) = (1,2)$?

$z = f(1,2) + f_x(1,2)(x-1) + f_y(1,2)(y-2)$

$= 3 - (x-1) + 4(y-2)$

Equation of tangent plane is $z = 3 - (x-1) + 4(y-2)$

or $z = -x + 4y - 4$

(b) What is the linearization of $f(x,y)$ at the point $(x,y) = (1,2)$?

The linearization is $f(x,y) \approx 3 - (x-1) + 4(y-2)$

$= -x + 4y - 4$

(c) Estimate $f(1.2, 1.9)$

$f(1.2, 1.9) \approx$ 2.4

$f(1.2, 1.9) \approx 3 - (1.2 - 1) + 4(1.9 - 2)$

$= 3 - (.2) + 4(-.1)$

$= 2.4$

9. (5 points) If $\mathbf{r}(t) = \langle x(t), y(t) \rangle$ the give the chain rule for $\frac{d}{dt} f(\mathbf{r}(t))$.

$$\frac{d}{dt} f(\mathbf{r}(t)) = \underline{\nabla f(\vec{r}(t)) \cdot \vec{r}\,'(t)}$$

10. (7 points) Let x and y be independent variables and assume that the dependent variable z is defined as a function of x and z

$$xz + y^2 z + z^3 = 4.$$

Use implicit differentiation to find $\frac{\partial z}{\partial y}$.

$$\frac{\partial z}{\partial y} = \underline{\frac{-2yz}{(x+y^2+3z^2)}}$$

$$\frac{\partial}{\partial y}(xz + y^2 z + z^3) = \frac{\partial}{\partial y}(4)$$

$$x \frac{\partial z}{\partial y} + 2yz + y^2 \frac{\partial z}{\partial y} + 3z^2 \frac{\partial z}{\partial y} = 0$$

$$(x + y^2 + 3z^2)\frac{\partial z}{\partial y} = -2yz$$

$$\frac{\partial z}{\partial y} = \frac{-2yz}{x+y^2+3z^2}$$

11. (7 points) Find the equation of the tangent plane to $x^2 + y^2 + z^2 = 14$ at the point $(1, -2, 3)$.

The equation is $\underline{2x - 4y + 6z = 28}$

If $f = x^2 + y^2 + z^2$

then $\nabla f(1, -2, 3)$ will be normal to the tangent plane.

$$\nabla f = \langle 2x, 2y, 2z \rangle$$

$$\nabla f(1, -2, 3) = \langle 2, -4, 6 \rangle$$

so the tangent plane is

$$\boxed{2(x-1) - 4(y-(-2)) + 6(z-3) = 0}$$

or $2x - 4y + 6z = 28$

This is also ok.

12. (5 points) Find the curvature of the curve

$$\mathbf{r}(t) = \langle \cos(t), \sin(t), t \rangle$$

$\kappa(t) =$ $\frac{1}{2}$

$$\kappa = \frac{|\vec{r}\,'(t) \times \vec{r}\,''(t)|}{|\vec{r}\,'(t)|^3}$$

$$\vec{r}\,'(t) = \langle -\sin(t), \cos(t), 1 \rangle$$

$$\vec{r}\,''(t) = \langle -\cos(t), -\sin(t), 0 \rangle$$

$$\vec{r}\,'(t) \times \vec{r}\,''(t) = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -\sin(t) & \cos(t) & 1 \\ -\cos(t) & -\sin t & 0 \end{vmatrix}$$

$$= \left\langle \begin{vmatrix} \cos t & 1 \\ -\sin t & 0 \end{vmatrix}, -\begin{vmatrix} -\sin t & 1 \\ -\cos t & 0 \end{vmatrix}, \begin{vmatrix} -\sin t & \cos t \\ -\cos t & -\sin t \end{vmatrix} \right\rangle$$

$$= \langle \sin t, -\cos t, 1 \rangle$$

$$|\vec{r}\,'(t)| = \sqrt{\sin^2 t + \cos^2 t + 1} = \sqrt{2}$$

$$|\vec{r}\,'(t) \times \vec{r}\,''(t)| = \sqrt{\sin^2 t + \cos^2 t + 1} = \sqrt{2}$$

$$\kappa = \frac{\sqrt{2}}{(\sqrt{2})^3} = \frac{1}{2}$$