Math 554, Test 3 Name: Answer Key.

1. (a) Let $f\colon E \to E'$ be a map between metric spaces and let $a \in E$. Give the ε-δ definition of what it means for f to be ***continuous*** at a.

SOLUTION: For all $\varepsilon > 0$ there is a $\delta > 0$ such that if $p \in E$ with $d(p, a) < \delta$, then $d'(f(p), f(a)) < \varepsilon$.

(b) Let $f\colon E \to E'$ be a map between metric spaces. Define what it means for f to be ***uniformly continuous***.

SOLUTION: For all $\varepsilon > 0$ there is a $\delta > 0$ such that for all $p, q \in E$, if $d(p, q) < \delta$, then $d'(f(p), f(q)) < \varepsilon$.

(c) Let $f\colon E \to \mathbf{R}$ and let $A \subseteq E$ be a nonemepty subset of A. Define what it means for f ***to achieve its maximum*** on A.

SOLUTION: There is a $p_0 \in A$ such that $f(a) \leq f(p_0)$ for all $a \in A$.

(d) Fill in the blanks. Let E be a metric space and $f\colon E \to \mathbf{R}$ a function. Then, if E is <u>compact</u> and f is <u>continuous</u> the function f achieves its its maximum and minimum.

(e) Let $\langle p_n \rangle_{n=1}^{\infty}$ is a sequence in the metric space E. Define $\lim_{n\to\infty} p_n = p$.

SOLUTION: For all $\varepsilon > 0$ there is a N such that if $n > N$, then $d(p_n, p) < \varepsilon$.

2. (a) Let $f\colon E \to E'$ be a map between metric spaces. State precisely what it means for the "preimage of open sets to be open".

SOLUTION: If $U \subseteq E'$ is open, then the preimage $f^{-1}[U] := \{x \in E : f(x) \in U\}$ is an open subset of E.

(b) Let $f\colon E \to E'$ and and $g\colon E' \to E''$ be continuous functions between metric spaces. Use that the preimage of open sets by continuous functions are open to prove that the composition $g \circ f$ is continuous. *Hint:* You are allowed to assume that for any subset $V \subseteq E''$ that $(g \circ f)^{-1}[V] = f^{-1}[g^{-1}[V]]$.

SOLUTION: We know that a function, h, between metric spaces is continuous if and only if the preimages under h of open sets are open.

Let $U \subseteq E''$ be open. Then the preimage $g^{-1}[U]$ is an open subset of E' as g is continuous. As f is continuous this implies $f^{-1}[g^{-1}[U]] = (f \circ g)^{-1}[U]$ is open in E. Therefore the preimage by $f \circ g$ of open sets is open and thus $f \circ g$ is continuous.

3. (a) Define what it means for the metric space E to be connected. (You only have to give one of the three equivalent definitions we gave.)

SOLUTION: The three equivalent definitions we gave were

(i) E is not the disjoint union of two nonempty open subsets of E.
(ii) E is not the disjoint union of two nonempty closed subsets of E.
(iii) The only subsets of E that are both open and closed are $\varnothing$ and E.

Any one of these is correct.

(b) If the metric space E is connected and the function $f\colon E \to E'$ is continuous prove that the image $f[E]$ is connected.

SOLUTION: Towards a contradiction assume $f[E]$ is not connected. Then $f[E]$ is the disjoint union of two nonempty open subsets of $f[E]$, say $f[E] = U \cup V$ with $U \cap V = \varnothing$, $U \cup V = f[E]$, $U, V \neq \varnothing$ and with U and V open in $f[E]$. By basic properties of preimages,

$$E = f^{-1}[f[E]] = f^{-1}[U \cup V] = f^{-1}[U] \cup f^{-1}[V],$$
$$f^{-1}[U] \cap f^{-1}[V] = f^{-1}[U \cap V] = f^{-1}[\varnothing] = \varnothing,$$

and $f^{-1}[U]$ and $f^{-1}[V]$ are both nonempty. Also, as f is continuous, the preimages $f^{-1}[U]$ and $f^{-1}[V]$ are open. Thus we have shown that $E = f^{-1}[U] \cup f^{-1}[V]$ is the disjoint union of nonempty open sets contradicting that E is connected.

4. (a) Carefully state the ***intermediate value theorem*** for a function $f\colon [a, b] \to \mathbf{R}$.

SOLUTION: Let $f\colon [a, b] \to \mathbf{R}$ be continuous. Then for any y between $f(a)$ and $f(b)$, there is an $x \in (a, b)$ with $f(x) = y$. (Note here that x is in the open interval (a, b) not just the closed interval.)

(b) Let $f(x) = x^3 - 4x + 2$. Show that $f(x) = 0$ has at least 3 solutions. *Hint:* What are the values of $f(-3)$, $f(0)$, $f(1)$ and $f(2)$?

SOLUTION: The function f is continuous as it is a polynomial. By direct calculation

$$f(-3) = -13, \qquad f(0) = 2, \qquad f(1) = -1, \qquad f(2) = 2.$$

The restriction of f to $[-3,0]$ is continuous and 0 is between $f(-3) = -13$ and $f(0) = 2$ thus there is an x_1 in the open interval $(-3,0)$ with $f(x_1) = 0$.

Likewise the restriction of f to $[0,1]$ is continuous and 0 is between $f(0) = 2$ and $f(1) = -1$ thus there is an x_2 in the open interval $(0,1)$ with $f(x_2) = 0$.

Finally the restriction of f to $[1,2]$ is continuous and 0 is between $f(1) = -1$ and $f(2) = 2$ thus there is an x_3 in the open interval $(1,2)$ with $f(x_3) = 0$.

Thus x_1, x_2, x_3 are solutions. They are also distinct as the intervals $(-3,0)$, $(0,1)$, and $(1,2)$ are disjoint. Thus there are at least three solutions.

REMARK: Some of you were a bit sloppy about showing that x_1, x_2, and x_3 are distinct. Note that the closed intervals $[-3,0]$ and $[0,1]$ are not disjoint, so if you only said that $x_1 \in [-3,0]$ and $x_2 \in [0,1]$ you still need to to a bit of arguing to show $x_1 \neq x_2$.

5. Let $f, g\colon E \to \mathbf{R}$ be continuous functions form a metric space E to $\mathbf{R}$. Prove that $f+g$ is continuous.

SOLUTION: Let $p \in E$. We will show that $f+g$ is continuous at p. As f is continuous there is a $\delta_1 > 0$ such that

$$d(x,p) < \delta_1 \qquad \implies \qquad |f(x) - f(p)| < \frac{\varepsilon}{2}.$$

As g is continuous there is a $\delta_2 > 0$ such that

$$d(x,p) < \delta_2 \qquad \implies \qquad |g(x) - g(p)| < \frac{\varepsilon}{2}.$$

Set $\delta = \min\{\delta_1, \delta_2\}$. Then if $d(x,p) < \delta$ we have

$$|(f+g)(x) - (f+g)(p)| \leq |f(x) - f(p)| + |g(x) - g(p)| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

This shows that $f+g$ is continuous at p. As p was any point of E, this shows that $f+g$ is continuous on all of E.

ALTERNATE SOLUTION: We can also prove this by using that a function between metric spaces is continuous if and only if it does the "right thing to sequences". That is a map $h\colon E \to E'$ between metric spaces is continuous if and only if for every convergent sequence, $\lim_{n\to\infty} p_n = p$, in E, we have that $\lim_{n\to\infty} h(p_n) = h(p)$.

In our case we have that the functions $f, g \to \mathbf{R}$ are continuous so if $\lim_{n\to\infty} p_n = p$ in E then the limits

$$\lim_{n\to\infty} f(p_n) = f(p), \qquad \text{and} \qquad \lim_{n\to\infty} g(p_n) = g(p).$$

But by properties of limits of real valued sequences we then have

$$\lim_{n\to\infty} (f(p_n) + g(p_n)) = f(p) + g(p).$$

This shows that $f + g$ does the right thing to sequences and therefore $f + g$ is continuous.