Math 554 Homework

Problem 1. From the text do Problems 4, 6, and 7 on page 61. If you find it interesting you can do Problem 5 as extra credit. $\square$

Before doing Problem 6, you may want to do the problems below as hint on how deal with the inequalities involved.

We are at the point where we need to come to grips with inequalities. In class today we looked at problems like this. Assume

$$|x-a|, |y-b| < r \tag{1}$$

then find an upper bound on

$$|(4x-7x+1)-(4x-7b+1)|.$$

And here is a way of doing this.

$$\begin{aligned} |(4x-7x+1)-(4a-7b+1)| &= |(4(x-a)-7(y-b)| \\ &\leq 4|x-a| + 7|y-b| \\ &< 4r + 7r \\ &= 11r \end{aligned} \tag{2}$$

Using this we now show

Proposition 1.

$$V = \{(x,y) : 4x - 7x \neq -1\}$$

is open an open subset of the plane $\mathbf{R}^2$.

Proof. First note we can rewrite V as

$$V = \{(x,y) : |4x-7x+1| > 0\}.$$

So we need to show that if $(a,b) \in V$, that there is an $r > 0$ such that

$$B((a,b),r) \subseteq V.$$

If $(x,y) \in B((a,b),r)$,

$$|x-a|, |y-b| < r \tag{3}$$

and so it is enough to show there exists an $r > 0$ such that

$$|x-a|, |y-b| < r \quad \Longrightarrow \quad |4x-7x+1| > 0.$$

We have already seen that the inequalities (3) implies

$$|(4x-7x+1)-(4a-7b+1)| < 11r. \tag{4}$$

Therefore we can use the adding and subtracting trick

$$\begin{aligned} |4x-7x+1| &= |(4a-7b+1)+(4x-7x+1)-(4a-7b+1)| \\ &\geq |(4a-7b+1)| - |(4x-7x+1)-(4a-7b+1)| && \text{(reverse triangle ineq.)} \\ &|(4a-7b+1)| - 11r && \text{(by (4))} \end{aligned}$$

So if we let

$$r = \frac{|(4a - 7b + 1)|}{11}$$

we then have

$$|4x - 7x + 1| > |(4a - 7b + 1)| - 11r = |(4a - 7b + 1)| - 11\frac{|(4a - 7b + 1)|}{11} = 0$$

which is just what we needed to get that

$$B((a, b), r) \subseteq V.$$

As (a, b) as an arbitrary point of V, this shows that V contains an open ball about any of its points and therefore is open. $\square$

We now consider more complicated functions than linear ones. Here is are some examples.

Problem 2. Show that if $|a|, |x| \leq 10$, then

$$|x^2 - a^2| \leq 20|x - a|.$$

Hint: $x^2 - a^2 = (x + a)(x - a)$.

Problem 3. Show that if $|a|, |x| \leq R$, then

$$|x^3 - a^3| \leq 3R^2|x - a|.$$

Hint: Can you factor $x^3 - a^3$?

Problem 4. If

$$|a| \leq 10, \qquad |x - a| < r, \qquad \text{and} \qquad r \leq 1$$

show

$$|x| \leq 11$$

and then use this to show

$$|x^2 - a^2| \leq 21|x - a|.$$

$\square$

Example. If $|a|, |b|, |x|, |y| \leq 20$, $|x - a| < r$, and $|y - b| < r$, then

$$|xy - ab| < 40r.$$

To see this we use the adding and subtracting trick

$$\begin{aligned}
|xy - ab| &= |xy - ay + ay - ab| \\
&= |(x - a)y + a(y - a)| \\
&\leq |x - a||y| + |a||y - a| && \text{(triangle inequality)} \\
&\leq 20|x - a| + 20|y - b| && \text{(as } |a|, |y| \leq 20\text{)} \\
&< 20r + 20r && \text{(as } |x - a|, |y - b| < r\text{)} \\
&= 40r.
\end{aligned}$$

$\square$

Problem 5. As a variant on the last example let

$$|a|, |b| \le 20, \quad |x-a|, |y-b| < r, \quad \text{and} \quad r \le 1.$$

Then show

$$|x|, |y| \le 21$$

and use this to show

$$|xy - ab| < 41r$$

□

Problem 6. We can generalize the last problem. Assume

$$|x-a|, |y-b| < r, \quad \text{and} \quad r \le 1.$$

Then show

$$|x| \le |a| + 1, \qquad \text{and} \qquad |y| \le |b| + 1$$

and then that

$$|xy - ab| < (|a| + |b| + 1)r.$$

□

Problem 7. Assume

$$|x-a|, |y-b| < r, \quad \text{and} \quad r \le 1.$$

Then

$$|xy - 1| \ge |ab - 1| - (|a| + |b| + 1)r.$$

Hint: One way is to start with the adding and subtracting trick and the reverse triangle inequality:

$$|xy| = |ab + xy - ab| \ge |ab| - |xy - ab|.$$

□