Math 554 Homework

There are some algebraic identities we will need during the term. One is that for any positive integer and all real numbers

$$x^n - y^n = (x-y)(x^{n-1} + x^{n-2}y + x^{n-3}y^2 + \cdots + y^{n-1}).$$

Let us check this for $n = 4$. We start with the right side and simplify.

$$\begin{aligned}(x-y)(x^3 + x^2y + xy^2 + y^3) &= x(x^3 + x^2y + xy^2 + y^3) - y(x^3 + x^2y + xy^2 + y^3)\\ &= x^4 + x^3y + x^2y^2 + xy^3 - x^3y - x^2y^2 - xy^3 - y^4\\ &= x^4 - y^4\end{aligned}$$

Problem 1. Prove that

$$x^n - y^n = (x-y)(x^{n-1} + x^{n-2}y + x^{n-3}y^2 + \cdots + y^{n-1})$$

for all positive integers n and all $x, y \in \mathbf{R}$. □

A related identity is that for all positive integers n and real numbers a, r with $r \neq 1$

$$a + ar + ar^2 + \cdots + ar^n = \frac{a - ar^{n+1}}{1-r}$$

Here is a proof when $n = 4$. Set

$$S = a + ar + ar^2 + ar^3 + ar^4.$$

Multiply by r

$$rS = ar + ar^2 + ar^3 + ar^4 + ar^5$$

Now subtract

$$\begin{aligned}(1-r)S = S - rS =& a + ar + ar^2 + ar^3 + ar^4\\ & - ar - ar^2 - ar^3 - ar^4 - ar^5\\ =& a - ar^5.\end{aligned}$$

As $(1-r) \neq 0$ we can divide by $(1-r)$ to get

$$S = \frac{a - ar^5}{1-r}.$$

Problem 2. Prove that for any positive integer n and any real numbers a, r with $r \neq 1$ that

$$a + ar + ar^2 + \cdots + ar^n = \frac{a - ar^{n+1}}{1-r}$$

holds. □

Problem 3. From the text for the problems set starting on page 29 do problems 2, 4a(do not use a calculator), 5, 6, 7.