Math 554 Homework

Here is a generalization of a definition we made in class.

Definition 1. Let $f\colon E \to E$ be a map from a metric space to itself. Then x_* is a ***fixed point*** of $f(x_*) = f(x_*)$. (That is if x_* is fixed by f.) □

You proved on the last homework that if $f\colon [a,b] \to [a,b]$ is a continuous map from a closed bounded interval to itself, then f has a fixed point.

Problem 1. Let $f\colon \mathbf{R} \to \mathbf{R}$ be continuous and such that

$$|f(x)| \le 42 + \frac{1}{2}|x|.$$

Show that f has a fixed point. *Hint:* Let $g(x) = x - f(x)$. We want to use the intermediate value theorem to show that $g(x)$ has a zero. What can you say about the signs of $g(100)$ and $g(-100)$? □

Remark 2. One of the most famous theorems in mathematics is the ***Brouwer fixed-point Theorem*** which says that if $\overline{B}(\mathbf{a}, r) = \{\mathbf{x} : \mathbf{x} \in \mathbf{R}^n, \|\mathbf{x} - \mathbf{a}\| \le r\}$ a closed ball in $\mathbf{R}^n$, then any continuous map $f\colon B(\mathbf{a}, r) \to B(\mathbf{a}, r)$ has a fixed point. One corollary of this is that if you have a map of Columbia SC here in the city, then there is a point of the map that is directly over the point it represents. Unfortunately there does not seem to be a an elementary proof of this result. □

Problem 2. Assuming that $\cos(x)$ is continuous show that

$$\cos(x^2) = \frac{1}{4 + x^2}$$

has infinitely many solutions. □

Proposition 3. *Let $f, g\colon E \to \mathbf{R}$ be two uniformly continuous real valued functions on a metric space E. Then $f + g$ is uniformly continuous.*

Problem 3. Prove this. □

Proposition 4. *Let $f, g\colon E \to \mathbf{R}$ be two uniformly continuous real valued functions on a metric space E. Assume that f and g are both bounded. Then the product fg is also uniformly continuous.*

Problem 4. Prove this. *Hint:* If $x, y \in E$ we can use the adding and subtracting trick to get

$$\begin{aligned} |f(x)g(x) - f(y)g(y)| &= |f(x)g(x) - f(x)g(y) + f(x)g(y) - f(y)g(y)| \\ &\le |f(x)||g(x) - f(y)| + |f(x) - f(y)||g(y)|. \end{aligned}$$

which may be useful. □

Problem 5. This problem is basically a lemma for the next problem. $f\colon \mathbf{R} \to \mathbf{R}$ be a uniformly continuous function. So for $\varepsilon > 0$ let $\delta > 0$ be such that

$$|x - y| < \delta \quad \implies \quad |f(x) - f(y)| < \varepsilon.$$

(a) Show that if $x_0, x_1, \ldots, x_n \in \mathbf{R}$ such that $|x_j - x_{j-1}| < \delta$ for $j = 1, 2, \ldots, n$, then
$$|f(x_n) - f(x_0)| < n\varepsilon.$$
Hint: Start by using the adding and subtracting trick:
$$\begin{aligned} |f(x_n) - f(x_0)| &= \big|(f(x_n) - f(x_{n-1})) + (f(x_{n-1}) - f(x_{n-2})) \\ &\qquad + \cdots + (f(x_2) - f(x_1)) + (f(x_1) - f(x_0))\big| \\ &\le |f(x_n) - f(x_{n-1})| + |f(x_{n-1}) - f(x_{n-2})| \\ &\qquad + \cdots + |f(x_2) - f(x_1)| + |f(x_1) - f(x_0)| \end{aligned}$$

(b) Show that if n is a positive integer, then
$$|x - y| < n\delta \quad \Longrightarrow \quad |f(x) - f(y)| < n\varepsilon.$$
Hint: For $j = 0, 1, \ldots, n$, let
$$x_j = x + j\frac{y - x}{n}.$$
Then $x_0 = x$, $x_n = x + (y - x) = y$, and
$$|x_j - x_{j-1}| = \left|\left(x + j\frac{y-x}{n}\right) - \left(x + (j-1)\frac{y-x}{n}\right)\right| = \left|\frac{y-x}{n}\right| < \frac{n\delta}{n} = \delta.$$
and now you can use part (a). □

Problem 6. Let f and g be the functions on $\mathbf{R}$ given by $f(x) = x$ and $g(x) = \cos(x)$. You may assume that f and g are both continuous. Show that the product
$$fg(x) = x\cos(x)$$
is not uniformly continuous. This shows that in Proposition 4 it is important that both f and g are bounded. *Hint:* Towards a contradiction assume that $h(x) = x\cos(x)$ is uniformly continuous. Let $\varepsilon = 1$. Then there is a $\delta > 0$ such that
$$|x - y| < \delta \quad \Longrightarrow \quad |h(x) - h(y)| < 1.$$
Now choose n such that $n\delta > \pi/2$. Then by Problem 5
$$|x - y| < n\delta \quad \Longrightarrow \quad |h(x) - h(y)| < n \cdot 1 = n.$$
Let k be an integer with $2\pi k > n$ and let $x = 2\pi k$ and $y = 2\pi k + \pi/2$ to get a contradiction. □

Our big result about uniform continuity is

Theorem 5. *Any continuous function on a compact metric space is uniformly continuous. (A bit more precisely if $f\colon E \to E'$ is a continuous map between metric spaces and E is compact, then f is uniformly continuous.)* □

Here is an application.

Proposition 6. *If $f\colon \mathbf{R} \to \mathbf{R}$ is continuous and periodic, say $f(x + 1) = f(x)$, then f is uniformly continuous.*

Problem 7. Prove this along the following lines. Let $\varepsilon > 0$.

(a) Show that for any integer k that $f(x-k) = f(x)$.

(b) Explain why there is a $\delta > 0$ such that if $a, b \in [0,2]$ then

$$|a-b| < \delta \qquad \implies \qquad |f(b) - f(a)| < \varepsilon.$$

(c) Explain why there is no loss of generality in assuming that $\delta \leq 1$. (In (a) just replace δ with $\min\{1, \delta\}$.)

(d) Now let $x, y \in \mathbf{R}$ with $|x-y| < \delta$. We want to show $|f(x) - f(y)| < \varepsilon$. By possibly relabeling we can assume $x \leq y$. Let $k = \lfloor x \rfloor$ be the greatest integer in x. Show that if $a = x - k$ and $b = y - k$ that $a, b \in [0,2]$ and $|f(x) - f(y)| = |f(a) - f(b)|$. Use this to finish the proof. □

We now get the the last big theorem of the term. First a definition that we have done in class.

Definition 7. Let E and E' be metric spaces. Let $\langle f_n \rangle_{n=1}^{\infty}$ be a sequence of functions $f_n \colon E \to E'$. Then this sequences ***converges uniformly*** to the function $f \colon E \to E'$ iff for all $\varepsilon > 0$ there is an N such that

$$n > N \qquad \implies \qquad \text{for all } x \in E \quad d'(f(x), f_n(x)) < \varepsilon.$$

□

Here the term "uniform" in the definition means that we can choose N to work for all $x \in E$. That is $N = N(\varepsilon)$ can be chosen to depend on ε and not on x.

Theorem 8. *If $\langle f_n \rangle_{n=1}^{\infty}$ is a sequence of functions from the metric space E to the metric space E', $\lim_{n\to\infty} f_n = f$ uniformly, and each f_n is continuous, then f is continuous.*

A more concise statement would be "The uniform limit of continuous functions is continuous".

Problem 8. Prove this along the following lines. Let $p \in E$ and we will show that f is continuous at p. As p is an arbitrary point of E this is enough to show that f is continuous on all of E. Let $\varepsilon > 0$. Then there is an N such that

$$n > N \qquad \implies d'(f_n(x), f(x)) < \frac{\varepsilon}{3}$$

for all $x \in E$. Now choose any $n_0 > N$.

(a) Show that for all $x \in E$ we have

$$d'(f(x), f(p)) \leq d'(f(x), f_{n_0}(x)) + d'(f_{n_0}(x), f_{n_0}(p)) + d'(f_{n_0}(p), f(p)).$$

(b) Explain why there is a $\delta > 0$ such that for $x \in E$

$$d(x,p) < \delta \qquad \implies \qquad d'(f_{n_0}(x), f_{n_0}(p)) < \frac{\varepsilon}{3}.$$

(c) Put the last two parts together to show for $x \in E$

$$d(x,p) < \delta \qquad \implies \qquad d'(f(x), f(p)) < \varepsilon$$

which shows f is continuous at p and finishes the proof. □

Here is a variant

Theorem 9. *The uniform limit of uniformly continuous functions is uniformly continuous.*

Problem 9. Make this precise and prove it. *Hint:* The proof is very much like the proof of the proof of Theorem 8. □