Math 554 Homework

Here are some review subjects that we will need. First there is the binomial theorem. Recall that we define the ***factorials*** by

$$0! = 1, \quad 1! = 1, \quad 2! = 1 \cdot 2, \quad 3! = 1 \cdot 2 \cdot 3, \qquad n! = 1 \cdot 2 \cdot 3 \cdots (n-1) \cdot n.$$

The ***binomial coefficients*** are for non-negative integers n, k with $k \leq n$ by

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}.$$

Problem 1. Show that the following hold.

(a) $\displaystyle \binom{n}{0} = \binom{n}{n} = 1.$

(b) $\displaystyle \binom{n}{1} = \binom{n}{n-1} = n.$

(c) $\displaystyle \binom{n}{2} = \binom{n}{n-2} = \frac{n(n-1)}{2}.$

□

Problem 2 (Pascal's Triangle property). Show that if $0 \leq k \leq n-1$ are integers then

$$\binom{n}{k} + \binom{n}{k+1} = \binom{n+1}{k+1}.$$

□

Note if we put the binomial coefficients in a table

$$\begin{array}{ccccccccccc}
 & & & & & \binom{1}{1} & & & & & \\
 & & & & \binom{1}{0} & & \binom{1}{1} & & & & \\
 & & & \binom{2}{0} & & \binom{2}{1} & & \binom{2}{2} & & & \\
 & & \binom{3}{0} & & \binom{3}{1} & & \binom{3}{2} & & \binom{3}{3} & & \\
 & \binom{4}{0} & & \binom{4}{1} & & \binom{4}{2} & & \binom{4}{3} & & \binom{4}{4} & \\
\binom{5}{0} & & \binom{5}{1} & & \binom{5}{2} & & \binom{5}{3} & & \binom{5}{4} & & \binom{5}{5}
\end{array}$$

that any entry is the sum of the two directly above, which is what exactly what the relation $\binom{n}{k} + \binom{n}{k+1} = \binom{n+1}{k+1}$ says. This can be used to compute

$\binom{n}{k}$ for small values of n. For example up to $n = 5$ the binomial coefficients are given in the following table.

$$\begin{array}{ccccccccccc}
 & & & & & 1 & & & & & \\
 & & & & 1 & & 1 & & & & \\
 & & & 1 & & 2 & & 1 & & & \\
 & & 1 & & 3 & & 3 & & 1 & & \\
 & 1 & & 4 & & 6 & & 4 & & 1 & \\
1 & & 5 & & 10 & & 10 & & 5 & & 1
\end{array}$$

The relation $\binom{n}{k} + \binom{n}{k+1} = \binom{n+1}{k+1}$ also lets use prove the binomial theorem:

Theorem 1. *For any positive integer n and $(x, y \in \mathbf{R}$*

$$(x+y)^n = \binom{n}{0}x^n + \binom{n}{1}x^{n-1}y + \binom{n}{2}x^{n-2}y^2 + \cdots + \binom{n}{n-1}xy^{n-1} + \binom{n}{n}y^n.$$

In summation notation this is

$$(x + y) = \sum_{k=0}^{n} \binom{n}{k} x^{n-k}y^k.$$

Problem 3. Use induction to prove this. *Hint:* : The base case $n = 1$ is just that $(x + y)^1 = x + y = \binom{1}{0}x^1y^0 + \binom{1}{1}x^0 0y^1$. Here is the induction step from $n = 4$ to $n = 5$. That is we assume that we know

$$(x + y)^4 = \binom{4}{0}x^4 + \binom{4}{1}x^3y + \binom{4}{2}x^2y^2 + \binom{4}{3}xy^3 + \binom{4}{4}y^4$$

Then

$$\begin{aligned}
(x+y)^5 &= (x+y)(x+y)^4 \\
&= (x+y)\left[\binom{4}{0}x^4 + \binom{4}{1}x^3y + \binom{4}{2}x^2y^2 + \binom{4}{3}xy^3 + \binom{4}{4}y^4\right] \\
&= x\left[\binom{4}{0}x^4 + \binom{4}{1}x^3y + \binom{4}{2}x^2y^2 + \binom{4}{3}xy^3 + \binom{4}{4}y^4\right] \\
&\quad + y\left[\binom{4}{0}x^4 + \binom{4}{1}x^3y + \binom{4}{2}x^2y^2 + \binom{4}{3}xy^3 + \binom{4}{4}y^4\right] \\
&= \binom{4}{0}x^5 + \binom{4}{1}x^4y + \binom{4}{2}x^3y^2 + \binom{4}{3}x^2y^3 + \binom{4}{4}xy^4 \\
&\quad + \binom{4}{0}x^4y + \binom{4}{1}x^3y^2 + \binom{4}{2}x^2y^3 + \binom{4}{3}xy^4 + \binom{4}{4}y^5 \\
&= \binom{4}{0}x^5 + \left[\binom{4}{0} + \binom{4}{1}\right]x^4y + \left[\binom{4}{1} + \binom{4}{2}\right]x^3y^2 \\
&\quad + \left[\binom{4}{2} + \binom{4}{3}\right]x^2y^3 + \left[\binom{4}{3} + \binom{4}{4}\right]xy^4 + \binom{4}{0}y^5 \\
&= \binom{4}{0}x^5 + \binom{5}{1}x^4y + \binom{5}{2}x^3y^2 + \binom{5}{3}x^2y^3 + \binom{5}{4}xy^4 + \binom{4}{0}y^5 \\
&= \binom{5}{0}x^5 + \binom{5}{1}x^4y + \binom{5}{2}x^3y^2 + \binom{5}{3}x^2y^3 + \binom{5}{4}xy^4 + \binom{5}{0}y^5
\end{aligned}$$

where at the last step we have used

$$\binom{4}{0}x^5 = 1x^5 = \binom{5}{0}x^5, \quad \text{and} \quad \binom{4}{4}y^5 = 1y^5 = \binom{5}{5}y^5.$$

Now you do the general induction step from n to $n+1$. $\square$.

Problem 4. Use induction to prove ***Bernoulli's inequality***: If $x > -1$, then for any positive integer n

$$(1+x)^n \geq 1 + nx.$$

Hint: The induction step starts as $(1+x)^{n+1} = (1+x)(1+x)^n$. $\square$

Problem 5. Let S be a set of positive real numbers such which is bounded above and let a be a positive real number. Let $aS = \{as : s \in S\}$. Prove

$$\sup(aS) = a\sup(S).$$

$\square$

On $\mathbf{R}^3$, the three dimensional Euclidan space, we define the ***inner product*** on by

$$\mathbf{x} \cdot \mathbf{y} = x_1y_1 + x_2y_2 = x_3y_3$$

where $\mathbf{x} = (x_1, x_2, x_3)$ and $\mathbf{y} = (y_1, y_2, y_3)$.

Proposition 2. *For all* $\mathbf{x}, \mathbf{y}, \mathbf{z} \in \mathbf{R}^3$ *and* $a, b \in \mathbf{R}$ *we have*

$$\begin{aligned} (a\mathbf{x} + b\mathbf{y}) \cdot \mathbf{z} &= a\mathbf{x} \cdot \mathbf{z} + b\mathbf{y} \cdot \mathbf{z} \\ \mathbf{x} \cdot \mathbf{y} &= \mathbf{y} \cdot \mathbf{x} \\ \mathbf{x} \cdot \mathbf{x} &\geq 0 \qquad \text{with equality if and only if } \mathbf{x} = \mathbf{0} \end{aligned}$$

Proof. We have seen this in our vector calculus classes. □

For $\mathbf{x} \in \mathbf{R}^3$ we define

$$\|x\| = \sqrt{\mathbf{x} \cdot \mathbf{x}}.$$

Thus $\mathbf{x} \cdot \mathbf{x} = \|\mathbf{x}\|^2$.

Lemma 3. *Let* $a, b, c \in \mathbf{R}$ *with* $a \neq 0$*. Assume that*

$$at^2 + bt + c \geq 0 \quad \text{for all } t \in \mathbf{R}.$$

Then

$$D = b^2 - 4ac \leq 0.$$

*(D is the **discriminant** of* $at^2 + bt + c$*.)*

Proof. We saw that this holds in class. □

Theorem 4 (Cauchy Schwartz Inequality)**.** *It* $\mathbf{x}, \mathbf{y} \in \mathbf{R}^3$*, then the inequality*

$$|\mathbf{x} \cdot \mathbf{y}| \leq \|\mathbf{x}\|\|\mathbf{y}\|$$

holds.

Proposition 5. *Show that for* $\mathbf{x}, \mathbf{y} \in \mathbf{R}^3$ *and* $a \in \mathbf{R}$ *that*

$$\|a\mathbf{x}\| = |a|\|\mathbf{x}\|$$

and

$$\|\mathbf{x} + \mathbf{y}\| \leq \|\mathbf{x}\| + \|\mathbf{y}\|.$$

Problem 6. Prove this. *Hint:* We outlined the proof of $\|\mathbf{x}+\mathbf{y}\| \leq \|\mathbf{x}\|+\|\mathbf{y}\|$ in class. □

Theorem 6. *For* $p, q \in \mathbf{R}^3$ *define*

$$d(p, q) := \|p - q\|.$$

Show that d *makes* $\mathbf{R}^3$ *into a matric space.*

Proof. Prove this. □ □

Problem 7. Prove this. *Hint:* For all $t \in \mathbf{R}$ there holds

$$0 \leq \|t\mathbf{x} + y\|^2 = (t\mathbf{x} + y) \cdot (t\mathbf{x} + y).$$

Expand this to get a quadratic polynomial and then use Lemma 3. □

Let (E, d) be a metric space, $p \in E$ and $r > 0$. Then the ***open ball*** with center p and radius r is

$$B(p, r) := \{x \in E : d(x, p) < r\}$$

and the ***closed ball*** with center p and radius r is

$$\overline{B}(p, r) := \{x \in E : d(x, p) \leq r\}.$$

Problem 8*.* Let (E, d) be a metric space, $p \in E$, $r > 0$, and $q \in B(p, r)$. Show

$$B\big(q, r - d(p, q)\big) \subseteq B(p, r).$$

□

Problem 9*.* Problem 1 on page 61 of the text.

Problem 10*.* Problem 2 on page 61 of the text (if you find this too tricky, you can treat it as extra credit).