Math 554 Homework

Problem 1. Let $b_1, b_2, b_3, \ldots$ be a sequence of real numbers with

$$\lim_{n\to\infty} b_n = b$$

with $b \neq 0$. Show that there is a N such that

$$n > N \qquad \Longrightarrow \qquad |b_n| > \frac{|b|}{2}.$$

Hint: Let $\varepsilon = |b|/2$ in the definition of $\lim_{n\to\infty} b_n = b$ and use both the adding and subtracting trick and the reverse triangle inequality. $\square$

Theorem 1. *Let $b_1, b_2, b_3, \ldots$ be a sequence of real numbers with*

$$\lim_{n\to\infty} b_n = b$$

and assume that $b, b_n \neq 0$ for any n. Then

$$\lim_{n\to\infty} \frac{1}{b_n} = \frac{1}{b}.$$

Problem 2. Prove this. *Hint:* Note

$$\left| \frac{1}{b} - \frac{1}{b_n} \right| = \frac{|b_n - b|}{|b||b_n|}.$$

By Problem 1 there is a N_1 such that if $n > N_1$, then $|b_n| > |b|/2$. Thus

$$n > N_1 \qquad \Longrightarrow \qquad \frac{|b_n - b|}{|b||b_n|} \geq \frac{|b_n - b|}{|b|(|b|/2)} = \frac{2|b_n - b|}{|b|^2}.$$

And you should be able to take it from here. $\square$

Theorem 2. *Let $a_1, a_2, a_3, \ldots$ and $b_1, b_2, b_3, \ldots$ be sequences of real numbers with*

$$\lim_{n\to\infty} a_n = a, \qquad \lim_{n\to\infty} b_n = b$$

with $b, b_n \neq 0$ for any n. Then

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \frac{a}{b}.$$

Problem 3. Prove this. *Hint:* You should be able to do this without having to do the "Let $\varepsilon > 0$ and then find N" process by use of theorems we have already proven. Note

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} a_n \left(\frac{1}{b_n} \right)$$

and use our theorems on the limits of products and limits of reciprocals. $\square$

Theorem 3. *Let $a_1, a_2, a_3, \dots$ be a sequence of real numbers with real number*

$$\lim_{n\to\infty} a_n = a.$$

Then for any positive integer k and any real number c

$$\lim_{n\to\infty} c(a_n)^k = ca^k.$$

Problem 4. Prove this. *Hint:* Note

$$\lim_{n\to\infty} c(a_n)^{k+1} = \lim_{n\to\infty} c(a_n)^k(a_k).$$

which suggests an induction proof. □

Recall that a polynomial is a function of the form

$$f(x) = c_k x^k + c_{k-1}x^{k-1} + \cdots + c_1 x + c_0$$

where $c_0, c_1, \dots, c_{k-1}, c_k$ are real numbers.

Theorem 4. *Let $f(x) = c_k x^k + c_{k-1}x^{k-1} + \cdots + c_1 x + c_0$ be a polynomial and $a_1, a_2, a_3, \dots$ be a sequence of real numbers with*

$$\lim_{n\to\infty} a_n = a.$$

Then

$$\lim_{n\to\infty} f(a_n) = f(a).$$

Problem 5. Prove this. *Hint:* One way is induction on k. The base case is $k = 0$, that $f(x) = c_0$ a constant where the result is clear. For the induction step assume we know that for all polynomials, $g(x)$, of degree at most k that

$$\lim_{n\to\infty} g(a_k) = g(a)$$

and let $f(x)$ have degree $k+1$. Then

$$f(x) = c_{k+1}x^{k+1} + c_k x^k + c_{k-1}x^{k-1} + \cdots c_1 x + c_0 = c_{k+1}x^{k+1} + g(x)$$

where

$$g(x) = c_k x^k + c_{k-1}x^{k-1} + \cdots c_1 x + c_0$$

is a polynomial of degree at most k. But we know from Theorem 3 above that $\lim_{n\to\infty} c_{k+1}x^{k+1} = c_{k+1}a^{k+1}$ so we use this, the theorem on the limit of sums, and the induction hypothesis. □

We have seen, as our first example of a limit, that

$$\lim_{n\to\infty} \frac{1}{n} = 0.$$

With the results we have know we can do many of the limits you saw in calculus. Here are some examples. First, as easy examples of Theorem 4,

$$\lim_{n\to\infty}\left(3 + 4\frac{1}{n^2}\right) = 3 + 4\cdot 0 = 3, \quad \lim_{n\to\infty}\left(4 + \frac{1}{n} + 5\frac{1}{n^2}\right) = 4 + 1\cdot 0 + 5\cdot 0 = 4.$$

We can use these to find the more complicated limit

$$\lim_{n\to\infty} \frac{3n^2+4}{4n^2+n+5} = \lim_{n\to\infty} \frac{3+4/n^2}{4+1/n+5/n^2} = \frac{3}{4}.$$

Problem 6. Use the methods here to compute the following limits.

(a) $\displaystyle\lim_{n\to\infty} \frac{4n^3+2n^2-5n+1}{8n^3+4n^2+9}$.

(b) $\displaystyle\lim_{n\to\infty} \frac{n^4+n-18}{n^5+6n^3+7}$. □

Finally here is a more challenging problem.

Problem 7. Let $a_1, a_2, a_3, \ldots$ be a sequence of positive real numbers with real number

$$\lim_{n\to\infty} a_n = a$$

with $a > 0$. Then

$$\lim_{n\to\infty} \sqrt{a_n} = \sqrt{a}.$$

Hint: As a start for any positive numbers x and y we have the algebric idenity (rationalizing the numerator)

$$\sqrt{x} - \sqrt{y} = \frac{(\sqrt{x}-\sqrt{y})(\sqrt{x}+\sqrt{y})}{(\sqrt{x}+\sqrt{y})} = \frac{x-y}{\sqrt{x}+\sqrt{y}}.$$ □