Math 554 Homework

We have proven that $\mathbf{R}$ with its usual metric is complete. The next step is

Theorem 1. *The metric space $\mathbf{R}^n$ is complete.*

Problem 1. Prove this for $n = 2$. *Hint:* Let $p_1, p_2, p_3, \ldots$ be a Cauchy sequence in $\mathbf{R}^n$. Then we write $p_k = (x_k, y_k)$. One way to proceed to note that for all k, ℓ we have

$$|x_k - x_\ell|,\ |y_k - y_\ell| \le \sqrt{(x_k - x_\ell)^2 + (y_k - y_\ell)^2} = \|p_k - p_\ell\|$$

STEP 1. Use this to show that both the sequences $\langle x_k \rangle_{k=1}^{\infty}$ and $\langle y_k \rangle_{k=1}^{\infty}$ are Cauchy.

STEP 2. Show both the sequences $\langle x_k \rangle_{k=1}^{\infty}$ and $\langle y_k \rangle_{k=1}^{\infty}$ converge, say

$$\lim_{k\to\infty} x_k = x, \qquad \lim_{k\to\infty} y_k = y.$$

Let $p = (x, y)$.

STEP 3. The inequality

$$\|p - p_k\| = \sqrt{(x - x_k)^2 + (y - y_k)^2} \le \sqrt{2} \max\left\{|x - x_k|, |y - y_k|\right\}$$

holds.

STEP 4. Finish the proof. □

Recall that if (E, d) is a metric space and $S \subset E$, then S is also a metric space with the same metric (recall the restriction of d from $E \times E$ to $S \times S$. That is for $s, t \in S$ we still use the distance $d(s, t)$ as we used in E.

Theorem 2. *Let (E, d) be a complete metric space and $S \subseteq E$. Then (S, d) is complete if and only if S is closed in E.*

Problem 2. Prove this. *Hint:* First assume that S is closed and let $\langle s_n \rangle_{n=1}^{\infty}$ be a Cauchy sequence in S. Then $\langle s_n \rangle_{n=1}^{\infty}$ is also a Cauchy in E and as E is complete the sequence converges to a point of E. Now show this point is in S (it might be useful to recall that a closed set contains its limit points).

Second assume that S is complete as a metric space and let p be a limit point of S. Then there is a sequence $\langle s_n \rangle_{n=1}^{\infty}$ that converges to p. This sequence will be a Cauchy sequence and therefore converge to a point of S. Show this point is p (and here is might be useful to recall that a set that contains all its limit points is closed). □

Definition 3. A sequence of real numbers $a_1, a_2, a_3, \ldots$ is ***monotone increasing*** iff $a_1 \le a_2 \le a_3 \le \cdots$. It is ***monotone decreasing*** iff $a_1 \ge a_2 \ge a_3 \ge \cdots$. It is ***monotone*** if it is either monotone increasing or monotone decreasing. □

Theorem 4. *(a) A monotone increasing sequence of real numbers that is bounded from above is convergent.*

(b) A monotone decreasing sequence of real numbers that is bounded from below is convergent.

Problem 3. Prove part (a) of this. *Hint:* Let the sequence be $a_1 \leq a_2 \leq a_3 \leq \cdots$. As it is bounded above

$$a = \sup\{a_1, a_2, a_3, a_4, \dots\}$$

exists. Let $\varepsilon > 0$. Show that there is a N such that $a - \varepsilon < a_N \leq a$ and that for this N if $n > N$ then $|a - a_n| < \varepsilon$. □

Proposition 5. *Every sequence $a_1, a_2, a_3, \dots$ of real numbers has a monotone subsequence.*

Problem 4. Prove this. *Hint:* This is problem 14 on page 62 of the text. See the hint there. □

Proposition 6. *Let $\langle p_n \rangle_{n=1}^{\infty}$ be a Cauchy sequence in the metric space E. Assume that some subsequence $\langle p_{n_k} \rangle_{k=1}^{\infty}$ converges. Then the original sequence $\langle p_n \rangle_{n=1}^{\infty}$ converges.*

Problem 5. Prove this. (Note we are *not* assuming that E is complete. This Proposition is useful in showing that spaces are complete as in the next problem.) □

Problem 6. Combine Proposition 5 and Theorem 4 to give another proof that the real numbers are complete. □