Math 554 Homework

To review a bit of what we talked about in class today: Let E be a metric space and $S \subseteq E$. Then if $\{V_i\}_{i\in I}$ is a collection of subsets of E, then $\{V_i\}_{i\in I}$ is a ***cover*** of S iff $S \subset \bigcup_{i\in I} V_i$. The collection $\{V_i\}_{i\in I}$ is an ***open cover*** of S iff it is a cover of S and each V_i is open.

Definition 1. The subset S of the metric space E is ***compact*** iff for every open cover $\{V_i\}_{i\in I}$ of S there is a finite collection $\{V_{i_1}, V_{i_2}, \ldots, V_{i_n}\} \subseteq \{V_i\}_{i\in I}$ with $S \subset V_{i_1} \cup V_{i_2} \cup \cdots \cup V_{i_n}$. □

Put more briefly and eloquently: The set S is compact iff every open cover of S has a finite subcover.

Problem 1. Show every finite subset of a metric space is compact. □

We have seen the following:

Proposition 2. *If E is a compact metric space, then any closed subset of S is compact.* □

There is a partial converse

Proposition 3. *If S is a compact subset of a metric space E, then S is closed in E.*

Problem 2. Prove this. *Hint:* It is easiest to prove the contrapositive: If S is not closed, then S is not compact. So assume that S is not closed. Then S has a limit point p with $p \notin S$. For any $r > 0$ let $V_r = \mathcal{C}\overline{B}(p,r)$ (that is V_r is the compliment of the closed ball $\overline{B}(p,r)$). Then show $\{V_r\}_{r>0}$ is an open cover of S. If $\{V_{r_1}, V_{r_2}, \ldots, V_{r_n}\}$ is a finite subset of $\{V_r\}_{r>0}$ then

$$\begin{aligned} V_{r_1} \cup V_{r_2} \cup \cdots \cup V_{r_n} &= \mathcal{C}\overline{B}(p,r_1) \cup \mathcal{C}\overline{B}(p,r_2) \cup \cdots \cup \mathcal{C}\overline{B}(p,r_n) \\ &= \mathcal{C}\Big(B(p,r_1) \cap B(p,r_2) \cap \cdots \cap B(p,r_n)\Big) \\ &= \mathcal{C}\overline{B}(p,r_*) \end{aligned}$$

where $r_* = \min\{r_1, r_2, \ldots, r_n\}$. Now use that p is a limit point to show this finite union can not cover S. □

Definition 4. Let E be a metric space and $S \subseteq E$. Then $p \in E$ is a ***cluster point*** of S iff every open ball, $B(p,r)$, about p contains infinitely points of S. □

Note that it is not required that the cluster point be in S. For example for the open interval (a,b) the set of cluster points is the closed interval $[a,b]$. Of these the points a and b are not in (a,b).

Problem 3. For the following sets, S, give the set of cluster points of the set and say which of these are in S.

(a) The open ball $S = B((0,0),r)$ of radius r about the origin in $\mathbf{R}^2$.

(b) The set $S = \mathbf{Q}$ of rational numbers in $\mathbf{R}$. $\square$

Theorem 5. *Show that every infinite subset of a compact metric space has a cluster point.*

Problem 4. Prove this. *Hint:* Let E be a compact metric space and $S \subseteq E$ an infinite set. Towards a contradiction assume that S does not have a cluster point. Then show for for each $p \in E$ there is a $r_p > 0$ such that B_{r_p} only contains a finite number of points of S. Show $\{B(p, r_p)\}_{p\in E}$ is an open cover of E. Now take a finite subcover and recall that a finite union of finite sets is a finite. $\square$

Proposition 6. *If E is a compact metric space, then every sequence $\langle p_n\rangle_{n=1}^{\infty}$ has a convergent subsequence.*

Problem 5. Prove this. *Hint:* Split this into two cases. Case 1. The set $\{p_1, p_2, \ldots\}$ is infinite. Then this set will have a cluster point, p. Show there is a subsequence of the sequence that converges to p. Case 2. The set $\{p_1, p_2, \ldots\}$ is finite (an example of this would be the sequence of real numbers $p_n = (-1)^n$ that only takes on two values). Then there is a subsequence $\langle p_{n_k}\rangle_{k=1}^{\infty}$ which is constant, say $p_{n_k} = p$ for all k. $\square$

Theorem 7. *Let E be a compact metric space and let $K_1, K_2, K_3, \ldots$ be a sequence of nonempty closed subsets of E that are nested in the sense that*

$$K_1 \supseteq K_2 \supseteq K_3 \supseteq K_4 \supseteq \cdots .$$

Then

$$\bigcap_{n=1}^{\infty} K_n \neq \varnothing.$$

Problem 6. Prove this. *Hint:* Towards a contradiction assume $\bigcap_{n=1}^{\infty} K_n = \varnothing$, and show that if $V_n = \mathcal{C}K_n$ (that is V_n is the compliment of K_n), then $\{V_n\}_{n=1}^{\infty}$ is an open cover of E and use that to get a contradiction. $\square$

Problem 7. Give an example where Theorem 7 does not hold when the subsets K_k of the compact space E is are not closed. *Hint:* Maybe the easiest case is $E = [0, 1]$ and the K_n's are appropriately chosen open intervals. $\square$

So far we do not have any nontrivial examples of compact set. Here is a basic example.

Theorem 8 (Heine-Borel Theorem)**.** *The closed unit interval $I = [0, 1]$ is compact.*

Problem 8. Prove this along the following lines. If it is false, there is an open cover $\mathcal{V} = \{V_i\}_{i\in J}$ such no finite subset of $\mathcal{V}$ covers $I = [0, 1]$.

(a) Split I into two the intervals $[0, 1/2]$ and $[1/2, 1]$ each of which has a length of $1/2$. Explain why at least one of these intervals can not be covered by a finite number of elements of $\mathcal{V}$.

(b) Let I_1 be an interval from part (a) that has length $1/2$ and can not be covered by a finite number of elements of $\mathcal{V}$. We split I_1 into two closed interval of length $1/4$ and for the same reason as in part (a) at least one of these two intervals can not covered by a finite number of elements of $\mathcal{V}$. Let this interval be I_2. Continuing in the manner we get a sequence of closed intervals $I_1, I_2, I_3, \ldots$ with

$$I_1 \supset I_2 \supset I_3 \supset I_4 \supset \cdots$$

where the length of I_n is $1/2^n$ and none of the I_n's can be covered by a finite number of elements of $\mathcal{V}$. Let p_n be the midpoint of I_n. Prove $\langle p_n \rangle_{n=1}^{\infty}$ is a Cauchy sequence.

(c) As $\mathbf{R}$ is complete we then have that the limit $p = \lim_{n\to\infty} p_n$ exists. Explain why:

(i) $p \in I_n$ for all n, and

(ii) $p \in [0,1]$. *Hint:* For both of these it is useful to recall that a closed set contians all its limit points.

(d) As $p \in [0,1]$ there is a $V_i \in \mathcal{V}$ such that $p \in V_i$. As V_i is open this implies that there is an open ball $B(p,r) = (p-r, p+r) \subseteq V_i$. Choose n such that $1/2^n < r$ and show for this n that

$$I_n \subset B(p,r) \subseteq V_i$$

and explain why this gives a contradiction. □

Proposition 9. *Any closed subset of* $[0,1]$ *is compact.*

Problem 9. Prove this. *Hint:* Use Theorem 8 and Proposition 2. □