Math 554 Homework

We have shown

Proposition 1. *The following are equivalent for a metric space E.*

(a) E is the disjoint union of two nonempty open subsets of E.
(b) E is the disjoint union of two nonempty open subsets of E.
(c) E has a subset $U \neq \varnothing$, $U \neq E$ that is both open and closed. □

Definition 2. A metric space E satisfying any of one of three equivalent conditions of Proposition 1 is ***disconnected***.

But of course what we are really interested in is when a space is connected. This is when it is not disconnected:

Definition 3. The metric space E is connected iff it is not the disjoint union of two nonempty open subsets. □

Proposition 4. *If the metric space E is the disjoint union of the two nonempty open sets U and V and A is connected subset of E, then either $A \subseteq U$ or $A \subseteq V$.*

Problem 1. Prove this along the lines outlined in class. □

Proposition 5. *Let E be a metric space with*

$$E = A \cup \bigcup_{i \in I} B_i$$

where A and each B_i is nonempty and connected and for all $i \in I$ we have $A \cap B_i \neq \varnothing$. Then E is connected.

Problem 2. Prove this along the lines outlined in classes. □

Here is a generalization of Proposition 5.

Proposition 6. *Let $E = \bigcup_{i \in I} B_i$ were each B_i is nonempty and connected. Assume that for any $i, j \in I$ there is a finite sequence $i = i_1, i_2, \ldots, i_n = j \in I$ such that*

$$B_i \cap B_{i+1} \neq \varnothing$$

for $i = 1, 2, \ldots, n-1$. Then E is connected.

Problem 3. Prove this. *Hint:* Towards a contradiction assume that is the disjoint union of the two nonempty open subsets U and V. Choose $i_0 \in I$. Then B_{i_0} will have a point in common with either U or V. Assume it has a point in common with U. Then by Proposition 5 $B_{i_0} \subseteq U$. Then for any other B_j, connect it to B_{i_0} be a chain such as in the statement of the proposition and you take it form there. □

So far our deepest result about on connected sets is

Theorem 7. *Any nonempty interval in $\mathbf{R}$ is connected.* □

Problem 4. Let $S \subset \mathbf{R}$ be a nonempty connected subset of $\mathbf{R}$ that is neither bounded above or below. Show $S = \mathbf{R}$. *Hint:* Use the last theorem. $\square$

We have started to talk about continuous function between metric spaces.

Definition 8. Let (E, d) and (E', d') be metric space and $f \colon E \to E'$ a map from E to E'. Then f is ***continuous at the point*** $a \in E$ iff for all $\varepsilon > 0$ there is a $\delta > 0$ such that

$$d(x, a) < \delta \qquad \text{implies} \qquad d'(f(x), f(a)) < \varepsilon.$$

$\square$

Mostly we will be working with functions that are continuous at all points of their domains so we give a name to these.

Definition 9. The function $f \colon E \to E'$ is ***continuous*** iff it is continuous at all points $a \in E$. $\square$

Recall that if $f \colon E \to E'$ and $U \subseteq E$, the the ***preimage*** of U by f is

$$f^{-1}[U] = \{x \in E : f(x) \in U\}.$$

That is it is the set of all the x in E that get mapped into U by f.

The following relates continuity of a function $f \colon E \to E'$ with the open sets of E and E'.

Theorem 10. *Let (E, d) and (E', d') be metric spaces and $f \colon E \to E'$, a function from E to E'. Then f is continuous if and only for every open set $U \subseteq E'$ the set $f^{-1}[U]$ is open in E.*

A loose restatement of this would be that $f \colon E \to E'$ is continuous if and only if the preimage of open sets are open.

Problem 5. Prove the last theorem along the following lines.

(a) Assume that $f \colon E \to E'$ is continuous. Then we wish to show that for any open set $U \subseteq E'$ the preimage $f^{-1}[U]$ is open in E. To be specific we will be done when we have shown that for each $a \in f^{-1}[U]$ there is an open ball about a that is contianed in $f_{-1}[U]$. So let $a \in f^{-1}[U]$.
  (i) Explain why $f(a) \in U$ and why there is an $\varepsilon > 0$ such that $B(f(a), \varepsilon) \subseteq U$.
  (ii) As f is continuous at a there is a $\delta > 0$ such that $d(x, a) < \delta$ implies $d'(f(x), f(a)) < \varepsilon$. Use this to show $B(a, \delta) \subseteq f^{-1}[U]$ which is what we needed to finish this part of the proof.

(b) Now assume that the preimage under f of open sets are open. Then we want to show that f is continuous. Explictly we need to show that for any $a \in E$ and $\varepsilon > 0$ there is a $\delta > 0$ such that $d(x, a) < \delta$ implies $d'(f(x), f(a)) < \varepsilon$. So let $a \in E$ and $\varepsilon > 0$.
  (i) Explain why the set $f^{-1}[B(f(a), \varepsilon)]$ is an open subset of E.
  (ii) Explain why there is a $\delta > 0$ such that $B(a, \delta) \subseteq f^{-1}[B(f(a), \varepsilon)]$.
  (iii) Show that the last step implies $d(x, a) < \delta$ implies $d'(f(x), f(a)) < \varepsilon$ which finishes the proof. $\square$