Math 554 Homework

Definition 1. Let $f\colon E \to E'$ be a map between metric space. Then f is ***bounded*** iff there is a point $q \in E'$ such that $f(x) \in B(q,r)$ for all $x \in E$. □

Proposition 2. *If $f\colon E \to E'$ is a continuous map and E is compact, then f is bounded.*

Problem 1. Prove this. *Hint:* Choose a point $q \in E'$ and show that $\mathcal{V} := \{f^{-1}[B(q,r)] : r > 0\}$ is an open cover of E. □

Lemma 3. *If $f\colon E \to E'$ is a map between sets and $U \subseteq E'$ then*

$$f^{-1}[\mathcal{C}U] = \mathcal{C}f^{-1}[U]$$

where $\mathcal{C}V$ is the compliment of the set V. (That is the preimage of the compliment is the compliment of the preimage.)

We probably did this on one of the first homeworks this term, but as review, let's do it again.

Problem 2. Prove this. □

Using this we can give another characterization of continuous functions between metric spaces.

Proposition 4. *Let $f\colon E \to E'$ be a map between metric space. Then f is continuous if and only $f^{-1}[F]$ is closed for every closed subset F of E'. (That is a function is continuous if and only the preimage of closed sets is closed.)*

Problem 3. Prove this. □

Problem 4. Let E be a metric space and $p_0 \in E$. Define $f\colon E \to \mathbf{R}$ by

$$f(x) = d(x, p_0).$$

Show f satisfies

$$|f(x) - f(y)| \le d(x,y)$$

and use this to show f is continuous. □

Definition 5. Let p_0 be a cluster point of the metric space E and $f\colon \{p_0\} \to E'$ a map of $\{p_0\}$ to the metric space E'. Then we say

$$\lim_{p \to p_0} f(p) = q$$

iff for all $\varepsilon > 0$ there is δ such that

$$0 < d(p,p_0) < \delta \quad \implies \quad d(f(p),q) < \varepsilon.$$

□

Proposition 6. *Limits in the sense of the last definition are unique. That is if*

$$\lim_{p \to p_0} f(p) = q \quad \textit{and} \quad \lim_{p \to p_0} f(p) = q'$$

then $q = q'$.

Problem 5. Prove this. □

We have proven a result that relates limits of sequences to continuity. Loosely it says that a function, f, is continuous at a point, p_0, if it does the right thing by all sequences that converge to p_0. More precisely

Theorem 7. *Let* $f \colon E \to E'$ *be a map between metric spaces. Then* f *is continuous at* $p_0 \in E$ *if and only if for all sequences* $\langle p_n \rangle_{n=1}^{\infty}$ *in* E

$$\lim_{n \to \infty} p_n = p_0 \quad \textit{implies} \quad \lim_{n \to \infty} f(p_n) = f(p_0).$$

□

This allows us to use the theorems we have proven about limits of sequence to prove theorems about continuous functions. Recall that we know the following:

Proposition 8. *If* $\langle a_n \rangle_{n=1}^{\infty}$ *and* $\langle b_n \rangle_{n=1}^{\infty}$ *are sequences of real numbers with*

$$\lim_{n \to \infty} a_n = a, \qquad \textit{and} \qquad \lim_{n \to \infty} b_n = b$$

then

(a) $\lim_{n \to \infty} (a_n + b_n) = a + b$,

(b) $\lim_{n \to \infty} (c_1 a_n + c_2 b_n) = c_1 a + c_2 b$ *for any constants* $c_1, c_2 \in \mathbf{R}$,

(c) $\lim_{n \to \infty} a_n b_n = ab$. □

Using this and Theorem 7 we can easily show that various combinations of continuous real valued functions are continuous. Here is an example:

Proposition 9. *If* $f, g \colon E \to \mathbf{R}$ *are real valued continuous functions on the metric space* E*, then the sum* $f + g$ *is continuous.*

Proof. We will show that $f+g$ is continuous at every point $p_0 \in E$. By Theorem 7 it is enough to show that for every sequence in E with $\lim_{n \to \infty} p_n = p_0$ we have

$$\lim_{n \to \infty} (f(p_n) + g(p_n)) = f(p_0) + g(p_0).$$

As f and g are continuous at p_0 Theorem 7 tell us that

$$\lim_{n \to \infty} f(p_n) = f(p_0) \qquad \text{and} \qquad \lim_{n \to \infty} g(p_n) = g(p_0).$$

Proposition 8 (with $a_n = f(p_n)$, $a = f(p_0)$, $b_n = g(p_n)$, and $b = g(p_0)$) then gives

$$\lim_{n \to \infty} (f(p_n) + g(p_n)) = f(p_0) + g(p_0)$$

as required to finish the proof. □

Proposition 10. *Let $f, g\colon E \to \mathbf{R}$ be continuous real valued functions on a metric space. Then*
(a) The product fg is continuous.
(b) For any constants $c_1, c_2 \in \mathbf{R}$ the function $c_1 f + c_2 g$ is continuous.

Problem 6. Prove this. $\square$

Note that letting $f = g$ in Part (a) of the last theorem shows that if f is continuous, then so it f^2. This can be extended.

Proposition 11. *Let $f\colon E \to \mathbf{R}$ be a continuous real valued function on E. Then for any positive integer n the n-th power, f^n, of f is continuous. More generally of $c_0, c_1, \ldots, c_n$ are constants then the function $g = c_n f^n + c_{n-1} f^{n-1} + \cdots + c_1 f + c_0$ is continuous.*

Problem 7. Prove this. *Hint:* As we mentioned in class, one way to do this is by induction on n. If it is true for n then for $n+1$ we write

$$\begin{aligned} g &= c_{n+1} f^{n+1} + c_n f^n + \cdots + c_1 f + c_0 \\ &= f(c_{n+1} f^n + c_n f^{n-1} + \cdots + c_1) + c_0 \\ &= fG + c_0 \end{aligned}$$

and use that $G = c_{n+1} f^n + c_n f^{n-1} + \cdots + c_1$ is continuous by the induction hypothesis. $\square$

Problem 8. (a) Show the subset $U = \{(x, y) : x^2 + xy + y^4 \geq 7\}$ is a closed subset of the plane $\mathbf{R}^2$.
(b) Show the subset $F = \{(x, y, z) : x^2 + xy^3 - 9z^4 < 5\}$ is an open subset of $\mathbf{R}^3$. $\square$

Theorem 12. *Let $f\colon E \to E'$ be a continuous map between metric space with E compact. Then the image $f[E]$ is compact.*

Informally: The continuous image of compact sets are compact.

Problem 9. Prove the last theorem along the following lines. We need to show that every open cover of $f[E]$ has a finite sub-cover. So let $\mathcal{V}$ be an open cover of $f[E]$. Set

$$\mathcal{V}^* = \{f^{-1}[V] : V \in \mathcal{V}\}.$$

(a) Explain why each $f^{-1}[V] \in \mathcal{V}^*$ is an open subset of E.
(b) Show $\mathcal{V}^*$ is an open cover of E. As E is compact this implies there is a finite set $\{V_1, V_2, \ldots, V_n\} \subseteq \mathcal{V}$ such that

$$E \subseteq f^{-1}[V_1] \cup f^{-1}[V_2] \cup \cdots \cup f^{-1}[V_n].$$

(c) Show $\{V_1, V_2, \ldots, V_n\}$ open cover of $f[E]$ and explain why this finishes the proof. $\square$

Recall that we know that a compact subset of a metric space is closed and also is bounded. Thus (and we have seen this before)

Proposition 13. *Let K be a compact subset of the real numbers* $\mathbf{R}$*. Then K contains its maximum and minimum. More explicitly*

$$\sup(K) \in K \qquad \textit{and} \qquad \inf(K) \in K.$$

Proof. As K is compact, it is bounded both from above and below. Therefore $\sup(K)$ and $\inf(K)$ exist. But K is closed (as it is compact) and therefore it contains both $\sup(K)$ and $\inf(K)$. □

The following is one of the more important existence theorems in analysis.

Theorem 14. *If E is a compact metric space and $f\colon E \to \mathbf{R}$ is continuous, then f achieves both its maximum and minimum. More explicitly there are $p, q \in E$ such that for all $x \in E$*

$$f(p) \leq f(x) \leq f(q).$$

Problem 10. Prove this. *Hint:* From Theorem 12 the set $K = f[E]$ is a compact subset of $\mathbf{R}$. By Proposition 13 we have $\inf(K) \in K$ and $\sup(K) \in K$. □

In the last theorem it is definitely important that E is compact:

Problem 11. Give an example of a continuous function $f\colon (0,1) \to \mathbf{R}$ such that f does not achieve either its maximum or its minimum on $(0,1)$. □