Math 554 Homework

Just to save talking having to talk about the two cases $a < b$ and $b < a$ let's make a definition the covers both:

Definition 1. If a and b are distinct real numbers, then x is ***between*** a and b if either of $a < x < b$ or $b < x < a$ holds. $\square$

Definition 2. If $f\colon E \to \mathbf{R}$ is a function, and $a, b \in E$, then we say that f ***achieves all the values between*** $f(a)$ and $f(b)$ iff for all y between $f(a)$ and $f(b)$ there is an $x \in E$ with $f(x) = y$. $\square$

Note that saying that f achieves all the values between $f(a)$ and $f(b)$ is saying that for any y between $f(a)$ and $f(b)$ we can solve the equation $f(x) = y$ for x.

Recall that we showed in class that if $f\colon E \to E'$ is a map and $U, V \subseteq E'$ satisfy $U \cap V = \varnothing$, then also $f^{-1}[U] \cap f^{-1}[V] = \varnothing$. You may use this in what follows.

Theorem 3. *If $f\colon E \to E'$ is a continuous map between metric space and E is connected, then so is the image $f[E]$. (That is the continuous image of a connected space is connected.)*

Problem 1. Prove this. *Hint:* Towards a contradiction assume that $f[E]$ is not connected. Then $f[E] = U \cup V$ where U and V are disjoint non-empty open subsets of $f[E]$. Use these to get a contradiction by showing that E is the disjoint non-empty open subsets. $\square$

Theorem 4 (Intermediate Theorem First Version)**.** *If $f\colon E \to \mathbf{R}$ is a continuous real valued function on the connected metric space E and $a, b \in E$. Then f achieves all the values between $f(a)$ and $f(b)$.*

Problem 2. Prove this. $\square$

Theorem 5 (Intermediate Theorem Second Version)**.** *If $f\colon [a, b] \to \mathbf{R}$ is a continuous function, them f takes on all the values between $f(a)$ and $f(b)$.*

Problem 3. Prove this. *Hint:* If you don't say that $[a, b]$ is connected at some point in the proof you will lose points. $\square$

Our next goal is to show that all polynomials of odd degree have at least one real root. As we discussed in class the basic idea is clear that $f(x) = x^n + x_{n-1}x^{n-1} + \cdots + a_1 x + a_0$ with n odd then for values of x where $|x|$ is very large, the lower order terms are over powered by x^n and so the sign of $f(x)$ will be the same as the sign of x^n when $|x|$ is large. When n is odd this means that $f(x)$ changes sign and thus the Intermediate Value Theorem can be used to show there is a root. The technical details on this are a little messy, so here it is split up into lots of little lemmata.[1] I hope this does not make the proof look more complicated than it is.

[1] The correct, or at least the pretentious, plural of "lemma" is "lemmata".

Lemma 6. *If* $\alpha, \beta \in \mathbf{R}$ *then*

$$\alpha + \beta \geq \alpha - |\beta|$$

More generally if $\alpha, \beta_1, \beta_2, \ldots, \beta_n \in \mathbf{R}$ *then*

$$\alpha + \beta_1 + \beta_2 + \cdots + \beta_n \geq \alpha - (|\beta_1| + |\beta_2| + \cdots + |\beta_n|)$$

Problem 4. Prove this. $\square$

Lemma 7. *If* $|x| \geq 1$ *then*

$$|x| \leq |x|^2 \leq |x|^3 \leq \cdots \leq |x|^n$$

and therefore

$$\frac{1}{|x|} \geq \frac{1}{|x|^2} \geq \cdots \geq \frac{1}{|x|^n}.$$

Problem 5. Prove this. $\square$

Lemma 8. *If* $a_0, a_1, \ldots, a_{n-1} \in \mathbf{R}$ *and* $|x| \geq 1$ *then*

$$1 + \frac{a_{n-1}}{|x|} + \frac{a_{n-2}}{|x|^2} + \cdots + \frac{a_1}{|x|^{n-1}} + \frac{a_0}{|x|^n} \geq 1 - \left(\frac{|a_{n-1}| + |a_{n-2}| + \cdots + |a_0|}{|x|} \right).$$

Problem 6. Prove this. $\square$

Lemma 9. *If* $a_0, a_1, \ldots, a_{n-1} \in \mathbf{R}$ *and*

$$|x| \geq \max\{1, 2(|a_{n-1}| + |a_{n-2}| + \cdots + |a_0|)\}$$

then

$$1 - \left(\frac{|a_{n-1}| + |a_{n-2}| + \cdots + |a_0|}{|x|} \right) \geq \frac{1}{2}.$$

In particular $1 - \left(\frac{|a_{n-1}|+|a_{n-2}|+\cdots+|a_0|}{|x|} \right)$ *is positive.*

Problem 7. Prove this. $\square$

Lemma 10. *Let*

$$f(x) = x^n + x_{n-1}x^{n-1} + \cdots + a_1 x + a_0$$

be a monic polynomial of degree n *and set*

$$r = \max\{1, 2(|a_{n-1}| + |a_{n-2}| + \cdots + |a_0|)\}.$$

(a) If $x \geq r$, *then* $f(x)$ *is positive.*
(b) If n *is odd and* $x \leq -r$ *then* $f(x)$ *is negative.*

Problem 8. Prove this. $\square$

Theorem 11. *Any real polynomial of odd degree has at least one real root.*

Problem 9. Prove this. $\square$

Problem 10. Let E be a metric space that is disconnected, say $E = U \cup V$ where U and V are non-empty disjoint open subsets of E. Define $f\colon E \to \mathbf{R}$ by

$$f(x) = \begin{cases} 0, & x \in U; \\ 1, & x \in V. \end{cases}$$

Show that f is continuous. Note that the image of f is $f[E] = \{0, 1\}$ which is not connected. This shows that if E is not connected then there is always a continuous real valued function on E that does not satisfy the Intermediate Value Theorem. □

Proposition 12. *If $f\colon [a,b] \to [a,b]$ is a continuous function on a closed interval to its self, then the equation $f(x) = x$ has a solution in $[a,b]$.*

Problem 11. Prove this. □