Math 554, Test 1 Name

1. (30 points) State the following

(a) The subset S of $\mathbf{R}$ has an *upper bound*.

if there is some x s.t. $x \geq s$ $\forall s \in S$

(b) That α is the *least upper bound* (also called *suppremum*) of the set S.

iff $\alpha \geq s$ $\forall s \in S$ and $\alpha \leq x$ for any upper bound x of S

(c) That (E, d) is a *metric space*.

if E is a nonempty set, and $d(p,q)$ with $p,q \in E$ satisfies the following

(1) $d(p,q) \geq 0$ $\forall p,q \in E$
(2) $d(p,q) = d(q,p)$ $\forall p,q \in E$
(3) $d(p,r) \leq d(p,q) + d(q,r)$ $\forall p,q,r \in E$
(4) $d(p,q) = 0 \iff p = q$ $\forall p,q \in E$

(d) The set U is an *open subset* of a the metric space (E, d).

if $\forall p \in E$, p is the center of some open ball belonging to U

(e) The *Cauchy Schwartz inequality*.

$|\vec{x} \cdot \vec{y}| \leq \|\vec{x}\| \|\vec{y}\|$

(f) The *binomial theorem*.

$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^{n-k} y^k$

where $\binom{n}{k} = \frac{n!}{k!(n-k)!}$

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2. (15 points) Show that for any real numbers a, b that

$$a^2 + 4ab + 5b^2 \geq 0$$

with equality if and only if $a = b = 0$.

Consider $a^2+4ab+5b^2$

$= (a^2+4ab+4b^2) - 4b^2 + 5b^2$

$= (a+2b)^2 + b^2$

we know that $(a+2b)^2 + b^2 \geq 0$ because any value squared is a positive, and the sum of positive values is positive as well.

Equality can only happen if $a=b=0$

$(a+2b)^2 + b^2 = 0$

can only occur if $(a+2b)^2 = 0$ and $b^2 = 0$

$\Rightarrow (a+2b) = 0 \qquad b = 0$

$a = 2b$ but $b = 0$

so $a = b = 0$, iff $a^2+4ab+5b^2 = 0$

Nice Job!

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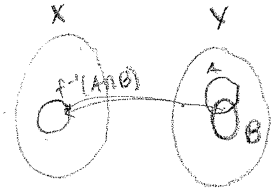
3. (15 points) Let $f\colon X \to Y$ be a function from the set X to the set Y, and let $A, B \subseteq Y$ be subsets of Y.

(a) Define the set $f^{-1}[A]$.

$$f^{-1}[A] = \{ x \in X : f(x) \in A \}.$$



(b) Prove

$$f^{-1}[A \cap B] = f^{-1}[A] \cap f^{-1}[B].$$

(1) Let $a \in f^{-1}[A \cap B] \Rightarrow a \in A \cap B$

$\Rightarrow a \in A$ and $a \in B$

$\Rightarrow a \in f^{-1}[A]$ and $a \in f^{-1}[B]$

$\Rightarrow a \in f^{-1}[A] \cap f^{-1}[B]$

So $f^{-1}[A \cap B] \subseteq f^{-1}[A] \cap f^{-1}[B]$.

(2) Let $a \in f^{-1}[A] \cap f^{-1}[B] \Rightarrow a \in f^{-1}[A]$ and $a \in f^{-1}[B]$.

$\Rightarrow a \in A$ and $a \in B$

$\Rightarrow a \in A \cap B$

$\Rightarrow a \in f^{-1}[A \cap B]$

Good job!

So $f^{-1}[A] \cap f^{-1}[B] \subseteq f^{-1}[A \cap B]$

Because $f^{-1}[A \cap B]$ and $f^{-1}[A] \cap f^{-1}[B]$ are subsets of each other, they are therefore equal

QED

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4. (15 points) (a) State the least ***least upper bound axiom***.

If a set S has an upper bond, then it has a least upper bond.

(b) Recall that Archimedes' principle is that for any real number x there is a positive integer n such that $n > x$. Prove Archimedes's principle from the least upper bound axiom.

Assume false, and we will arrive at a contradiction. So, $\exists x \in \mathbb{R}$ such that $x \geq n$, $\forall n \in \{1, 2, \ldots\}$. So, x is an upper bound for the set $\mathbb{N} = \{1, 2, 3, \ldots\}$. Now, by the l.u.b. axiom, since $\mathbb{N}$ has an upper bond, it has a leasy upper bonel. Let $x_0 = \sup(\mathbb{N})$. This means that $x_0 \geq n$ $\forall n \in \mathbb{N}$, and there is no upper bond for $\mathbb{N}$ that is less than x_0. Now, note that $n \in \mathbb{N} \Rightarrow n+1 \in \mathbb{N}$. So, $n+1 \leq x_0$ $\forall n \in \mathbb{N} \Rightarrow n \leq x_0 - 1$ $\forall n \in \mathbb{N}$. So, $x_0 - 1$ is an upper bond for $\mathbb{N}$. But $x_0 - 1 < x_0$, so x_0 is no longer a least upper bond. We have shown a contradiction, so we have proven that $\forall x \in \mathbb{R}$, $\exists n \in \mathbb{N}$ such that $n > x$.

NICE JOB!

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5. (15 points) Let $S \subseteq \mathbf{R}$ be a nonempty of real numbers that is bounded above. Let $2S = \{2s : s \in S\}$. Prove that

$$\sup(2S) = 2\sup(S).$$

$Sup(S) \geq s \ \forall s \in S$

So $2Sup(S) \geq 2s \ \forall s \in S$

So $2Sup(S) \geq Sup(2S)$ since $2Sup(S)$ is an upper bound for $Sup(2S)$.

Now to show $Sup(2S) \geq 2Sup(S)$.

We know that for $\varepsilon > 0$,

$\exists g \in S$ s.t. $Sup(S) < g + \varepsilon$.

So add this to itself to get

$2Sup(S) < 2(g + \varepsilon)$

$\Rightarrow 2Sup(S) < 2g + 2\varepsilon$

$\Rightarrow 2Sup(S) - 2\varepsilon < 2g \leq Sup(2S)$

but ε was arbitrary, so

$2Sup(S) \leq Sup(2S)$

Since both inequalities hold,

$2Sup(S) = Sup(2S)$

Good Job!

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6. (10 points) If $0 < x < 1$ show $x^3 < x^2 < x$.

$0 < x < 1$

$x < 1$

multiply by $x^2 \Rightarrow x \cdot x^2 < 1 \cdot x^2$

this is true because we know $x^2 > 0$ $\quad \underline{x^3 < x^2}$

now consider $x < 1$ and because $x > 0$, we can multiply both sides by x and it will, hold true.

$x \cdot x < 1 \cdot x$

$\underline{x^2 < x}$.

Since we have shown $x^3 < x^2$ and $x^2 < x$ we know that $x^3 < x^2 < x$.

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