Mathematics 172 Homework

Consider an organism with a three stage life history summarized by the loop diagram of Figure 1.

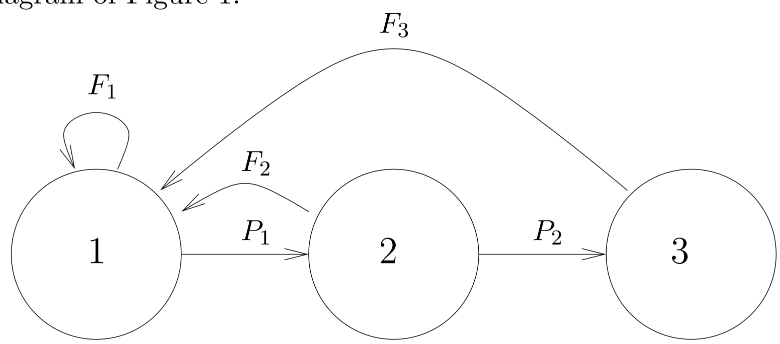

FIGURE 1

The Leslie matrix is

$$L = \begin{bmatrix} F_1 & F_2 & F_3 \\ P_1 & 0 & 0 \\ 0 & P_2 & 0 \end{bmatrix}$$

Our goal is to find a method compute the discrete growth rate, λ, the per capita growth rate, $r = \lambda - 1$ and the stable age distribution exactly. Let

$$\vec{n}(t) = \begin{bmatrix} n_1(t) \\ n_2(t) \\ n_3(t) \end{bmatrix}$$

If we have reached the stable age distribution then

$$\vec{n}(t+1) = \lambda \vec{n}(t)$$

that is

$$\begin{bmatrix} n_1(t+1) \\ n_2(t+1) \\ n_3(t+1) \end{bmatrix} = \lambda \begin{bmatrix} n_1(t) \\ n_2(t) \\ n_3(t) \end{bmatrix}$$

which leads to the three equations

$$\begin{aligned} n_1(t+1) &= \lambda n_1(t) \\ n_2(t+1) &= \lambda n_2(t) \\ n_3(t+1) &= \lambda n_3(t). \end{aligned}$$

From the loop diagram we have

$$\begin{aligned} n_1(t+1) &= F_1 n_1(t) + F_2 n_2(t) + F_3 n_3(t) \\ n_2(t+1) &= P_1 n_1(t) \\ n_3(t+1) &= P_2 n_2(t). \end{aligned}$$

This leads to

$$\lambda n_1(t) = F_1 n_1(t) + F_2 n_2(t) + F_3 n_3(t) \tag{1}$$
$$\lambda n_2(t) = P_1 n_1(t) \tag{2}$$
$$\lambda n_3(t) = P_2 n_2(t). \tag{3}$$

From equation (3) we get

$$n_3(t) = \frac{P_2}{\lambda} n_2(t).$$

From (2) we find

$$n_2(t) = \frac{P_1}{\lambda} n_1(t)$$

Combining these gives

$$n_3(t) = \frac{P_2}{\lambda} n_2(t) = \frac{P_2}{\lambda}\frac{P_1}{\lambda} n_1(t) = \frac{P_2 P_1}{\lambda^2} n_1(t).$$

Using our formulas for $n_2(t)$ and $n_3(t)$ in equation (1) give

$$\lambda n_1(t) = F_1 n_1(t) + F_2 \frac{P_1}{\lambda} n_1(t) + F_3 \frac{P_2 P_1}{\lambda^2} n_1(t).$$

Dividing by $\lambda n_1(t)$ we find that the $n_1(t)$ factor cancels out and the result is

$$1 = \frac{F_1}{\lambda} + \frac{F_2 P_1}{\lambda^2} + \frac{F_3 P_2 P_1}{\lambda^3}.$$

With a little more work this gives

Proposition 1. *The discrete grow rate λ is the unique positive solution to the **Euler-Lotka equation***

$$1 - \frac{F_1}{\lambda} - \frac{F_2 P_1}{\lambda^2} - \frac{F_3 P_2 P_1}{\lambda^3} = 0$$

which is easily solve on a calculator or computer. Then we also know the per capita growth rate $r = \lambda - 1$.

One we know λ we then have

$$n_2(t) = \frac{P_1}{\lambda} n_1(t), \qquad n_3(t) = \frac{P_2 P_1}{\lambda^2} n_1(t)$$

so that

$$\vec{n}(t) = \begin{bmatrix} n_1(t) \\ n_2(t) \\ n_3(t) \end{bmatrix} = \begin{bmatrix} n_1(t) \\ \frac{P_1}{\lambda} n_1(t) \\ \frac{P_2 P_1}{\lambda^2} n_1(t) \end{bmatrix} = n_1(t) \begin{bmatrix} 1 \\ \frac{P_1}{\lambda} \\ \frac{P_2 P_1}{\lambda^2} \end{bmatrix}$$

Which lets us compute the stable age distribution.

$$\text{Proportion in stage 1} = \frac{1}{1 + \dfrac{P_1}{\lambda} + \dfrac{P_2 P_1}{\lambda^2}} = \frac{\lambda^2}{\lambda^2 + P_1\lambda + P_2 P_1}$$

$$\text{Proportion in stage 2} = \frac{\dfrac{P_1}{\lambda}}{1 + \dfrac{P_1}{\lambda} + \dfrac{P_2 P_1}{\lambda^2}} = \frac{P_1\lambda}{\lambda^2 + P_1\lambda + P_2 P_1}$$

$$\text{Proportion in stage 3} = \frac{\dfrac{P_2 P_1}{\lambda^2}}{1 + \dfrac{P_1}{\lambda} + \dfrac{P_2 P_1}{\lambda^2}} = \frac{P_1 P_2}{\lambda^2 + P_1\lambda + P_2 P_1}$$

1. Use these formula to find the per capita growth rate, r, and the stable age distribution for the Leslie matrix

$$L = \begin{bmatrix} 0.05 & 1.3 & 10.7 \\ 0.16 & 0. & 0.0 \\ 0.0 & 0.55 & 0.0 \end{bmatrix}$$

Answer: $r = 068832$

$$\begin{aligned} \text{Proportion in sage 1 } &= 0.81517 \\ \text{Proportion in sage 2 } &= 0.13042 \\ \text{Proportion in sage 3 } &= 0.07173 \end{aligned}$$

2. Do the same of the Leslie matrix:

$$L = \begin{bmatrix} 0.15 & 1.3 & 5.4 \\ 0.2 & 0. & 0.0 \\ 0. & 0.8 & 0. \end{bmatrix}$$

Answer: $r = .068832$

$$\begin{aligned} \text{Proportion in sage 1} &= 0.7610 \\ \text{Proportion in sage 2} &= 0.1522 \\ \text{Proportion in sage 3} &= 0.1217 \end{aligned}$$